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Contents

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A view on separation axioms in an ordered intuitionistic fuzzy bitopological space

R. Narmada Devi[†], E. Roja[‡] and M. K. Uma[♯]

Department of Mathematics, Sri Sarada College for Women,
Salem, Tamil Nadu, India
E-mail: narmadadevi23@gmail.com

Abstract In this paper we introduce the concept of a new class of an ordered intuitionistic fuzzy bitopological spaces. Besides giving some interesting properties of these spaces. We also prove analogues of Uryshon's lemma and Tietze extension theorem in an ordered intuitionistic fuzzy bitopological spaces.

Keywords Ordered intuitionistic fuzzy bitopological space, lower (resp. upper) pairwise int -uitionistic fuzzy G_δ-α-locally T_1-ordered space, pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space, pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space, weakly pairwise in -tuitionistic fuzzy G_δ-α-locally T_2-ordered space, almost pairwise intuitionistic fuzzy G_δ-α-lo -cally T_2-ordered space and strongly pairwise intuitionistic fuzzy G_δ-α-locally normally order -ed space.
2000 Mathematics Subject Classification: 54A40, 03E72.

§1. Introduction

The concept of fuzzy sets was introduced by Zadeh [12]. Fuzzy sets have applications in many fields such as information [10] and control [11]. The theory of fuzzy topological spaces was introduced and developed by Chang [6]. The concept of fuzzy normal space was introduced by Bruce Hutton [5]. Atanassov [1] introduced and studied intuitionistic fuzzy sets. On the otherhand, Coker [7] introduced the notions of an intuitionistic fuzzy topological space and some other concepts. The concept of an intuitionistic fuzzy α-closed set was introduced by Biljana Krsteshka and Erdal Ekici [4]. G. Balasubramanian [2] was introduced the concept of fuzzy G_δ set. Ganster and Relly used locally closed sets [8] to define LC-continuity and LC-irresoluteness. The concept of an ordered fuzzy topological spaces was introduced and developed by A. K. Katsaras [9]. Later G. Balasubmanian [3] was introduced and studied the concepts of an ordered L-fuzzy bitopological spaces. In this paper we introduced the concepts of pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space, pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space, weakly pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space, almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space and strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space are introduced. Some interesting propositions are discussed. Urysohn's

lemma and Tietze extension theorem of an strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space are studied and estabilished.

§2. Preliminaries

Definition 2.1.[7] Let X be a nonempty fixed set and I is the closed interval [0, 1]. An intuitionistic fuzzy set (IFS) A is an object having the form $A = \{\langle x,\ \mu_A(x),\ \gamma_A(x)\rangle :\ x \in X\}$, where the mapping $\mu_A :\ X \longrightarrow I$ and $\gamma_A :\ X \longrightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously, every fuzzy set A on a nonempty set X is an IFS of the following form, $A = \{\langle x,\ \mu_A(x),\ 1 - \mu_A(x)\rangle :\ x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x,\ \mu_A(x),\ \gamma_A(x)\rangle :\ x \in X\}$.

Definition 2.2.[7] Let X be a nonempty set and the $IFSs$ A and B in the form $A = \{\langle x,\ \mu_A(x),\ \gamma_A(x)\rangle :\ x \in X\}$, $B = \{\langle x,\ \mu_B(x),\ \gamma_B(x)\rangle :\ x \in X\}$. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,

(ii) $\overline{A} = \{\langle x,\ \gamma_A(x),\ \mu_A(x)\rangle :\ x \in X\}$.

Definition 2.3.[7] The $IFSs$ $0_\sim$ and $1_\sim$ are defined by $0_\sim = \{\langle x,\ 0,\ 1\rangle :\ x \in X\}$ and $1_\sim = \{\langle x,\ 1,\ 0\rangle :\ x \in X\}$.

Definition 2.4.[7] An intuitionistic fuzzy topology (IFT) on a nonempty set X is a family τ of $IFSs$ in X satisfying the following axioms:

(i) $0_\sim,\ 1_\sim \in \tau$,

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1,\ G_2 \in \tau$,

(iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \tau$.

In this case the ordered pair $(X,\ \tau)$ or simply by X is called an intuitionistic fuzzy topological space ($IFTS$) on X and each IFS in τ is called an intuitionistic fuzzy open set ($IFOS$). The complement $\overline{A}$ of an $IFOS$ A in X is called an intuitionistic fuzzy closed set ($IFCS$) in X.

Definition 2.5.[7] Let A be an IFS in $IFTS$ X. Then

$int(A) = \bigcup\{G \mid G$ is an $IFOS$ in X and $G \subseteq A\}$ is called an intuitionistic fuzzy interior of A;

$clA = \bigcap\{G \mid G$ is an $IFCS$ in X and $G \supseteq A\}$ is called an intuitionistic fuzzy closure of A.

Definition 2.6.[4] Let A be an IFS of an $IFTS$ X. Then A is called an intuitionistic fuzzy α-open set ($IF\alpha OS$) if $A \subseteq int(cl(int(A)))$. The complement of an intuitionistic fuzzy α-open set is called an intuitionistic fuzzy α-closed set ($IF\alpha CS$).

Definition 2.7.[7] Let $(X,\ \tau)$ and $(Y,\ \phi)$ be two $IFTSs$ and let $f :\ X \to Y$ be a function. Then f is said to be fuzzy continuous iff the preimage of each IFS in ϕ is an IFS in τ.

Definition 2.8.[3] A L-fuzzy set μ in a fuzzy topological space X is called a neighbourhood of a point $x \in X$, if there exists an L-fuzzy set μ_1 with $\mu_1 \leq \mu$ and $\mu_1(x) > 0$. It can be shown that a L-fuzzy set μ is open $\Longleftrightarrow$ μ is a neighbourhood of each $x \in X$ for which $\mu(x) > 0$.

Definition 2.9.[3] The L-fuzzy real line $R(L)$ is the set of all monotone decreasing elements $\lambda \in L^R$ satisfying $\vee\{\lambda(t) \mid t \in R\} = 1$ and $\wedge\{\lambda(t) \mid t \in R\} = 0$, after the identification of

$\lambda,\ \mu \in L^R$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for all $t \in R$ where $\lambda(t-) = \wedge\{\lambda(s) \mid s < t\}$ and $\lambda(t+) = \vee\{\lambda(s) \mid s > t\}$.

Definition 2.10.[3] The natural L-fuzzy topology on $R(L)$ is generated from the subbasis $\{L_t,\ R_t \mid t \in R\}$, where $L_t[\lambda]=\lambda(t-)'$ and $R_t[\lambda]=\lambda(t+)'$.

Definition 2.11.[3] A partial order on $R(L)$ is defined by $[\lambda] \leq [\mu] \Leftrightarrow \lambda(t-) \leq \mu(t-)$ and $\lambda(t+) \leq \mu(t+)$ for all $t \in R$.

Definition 2.12.[3] The L-fuzzy unit interval $I(L)$ is a subset of $R(L)$ such that $[\lambda] \in I(L)$ if $\lambda(t) = 1$ for $0 < t$ and $\lambda(t) = 0$ for $t > 1$. It is equipped with the subspace L-fuzzy topology.

Definition 2.13.[3] Let $(X,\ \tau)$ be an L-fuzzy topological space. A function $f:\ X \to R(L)$ is called lower (upper) semicontinuous if $f^{-1}(R_t)(f^{-1}(L_t))$ is open for each $t \in R$. Equivalently f is lower (upper) semicontinuous $\Leftrightarrow$ it is continuous w. r. t the right hand (left hand) L-fuzzy topology on $R(L)$ where the right hand (left hand) topology is generated from the basis $\{R_t \mid t \in R\}(\{L_t \mid t \in R\})$. Lower and upper semi continuous with values in $I(L)$ are defined in the analogous way.

Definition 2.14.[3] A L-fuzzy set λ in a partially ordered set X is called

(i) Increasing if $x \leq y \Longrightarrow \lambda(x) \leq \lambda(y)$,

(ii) Decreasing if $x \leq y \Longrightarrow \lambda(x) \geq \lambda(y)$.

Definition 2.15.[2] Let ($X,\ T$) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called fuzy G_δ if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$ where each $\lambda_i \in T$. The complement of fuzzy G_δ is fuzzy F_σ.

Definition 2.16.[8] A subet A of a space $(X,\ \tau)$ is called locally closed (briefly lc) if $A = C \cap D$, where C is open and D is closed in $(X,\ \tau)$.

§3. Ordered intuitionistic fuzzy G_δ-α-locally bitopological spaces

In this section, the concepts of an intuitionistic fuzzy G_δ set, intuitionistic fuzzy α-closed set, intuitionistic fuzzy G_δ-α-locally closed set, upper pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space, lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space, pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space, pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space, weakly pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space, almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space and strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space are introduced. Some interesting propositions and characterizations are discussed. Urysohn's lemma and Tietze extension theorem of an strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space are studied and estabilished.

Definition 3.1. Let $(X,\ T)$ be an intuitionistic fuzzy topological space. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ be an intuitionstic fuzzy set of an intuitionistic fuzzy topological space X. Then A is said to be an intuitionistic fuzzy G_δ set (in short, IFG_δS) if $A = \bigcap_{i=1}^{\infty} A_i$, where each $A_i \in T$ and $A_i = \langle x,\ \mu_{A_i},\ \gamma_{A_i}\rangle$.

The complement of intuitionistic fuzzy G_δ set is said to be an intuitionistic fuzzy F_σ set (in short, IFF_σS).

Definition 3.2. Let $(X,\ T)$ be an intuitionistic fuzzy topological space. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ be an intuitionistic fuzzy set on an intuitionistic fuzzy topological space $(X,\ T)$. Then A is said be an intuitionistic fuzzy G_δ-α-locally closed set (in short, IFG_δ-α-lcs) if $A = B \cap C$, where B is an intuitionistic fuzzy G_δ set and C is an intuitionistic fuzzy α-closed set.

The complement of an intuitionistic fuzzy G_δ-α-locally closed set is said to be an intuitionistic fuzzy G_δ-α-locally open set (in short, IFG_δ-α-los).

Definition 3.3. Let $(X,\ T)$ be an intuitionistic fuzzy topological space. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space $(X,\ T)$. The intuitionistic fuzzy G_δ-α-locally closure of A is denoted and defined by $IFG_\delta\text{-}\alpha\text{-}lcl(A)=\bigcap\{B$: $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ is an intuitionistic fuzzy G_δ-α-locally closed set in X and $A \subseteq B\}$.

Definition 3.4. Let $(X,\ T)$ be an intuitionistic fuzzy topological space. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ be an intuitionistic fuzzy set in an intuitionistic fuzzy topological space $(X,\ T)$. The intuitionistic fuzzy G_δ-α-locally interior of A is denoted and defined by $IFG_\delta\text{-}\alpha\text{-}lint(A)=\bigcup\{B$: $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ is an intuitionistic fuzzy G_δ-α-locally open set in X and $B \subseteq A\}$.

Definition 3.5. An intuitionistic fuzzy set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ in an intuitionistic fuzzy topological space $(X,\ T)$ is said to be an intuitionistic fuzzy neighbourhood of a point $x \in X$, if there exists an intuitionistic fuzzy open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ with $B \subseteq A$ and $B(x) \supseteq 0_\sim$.

Definition 3.6. An intuitionistic fuzzy set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ in an intuitionistic fuzzy topological space $(X,\ T)$ is said to be an intuitionistic fuzzy G_δ-α-locally neighbourhood of a point $x \in X$, if there exists an intuitionistic fuzzy G_δ-α-locally open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ with $B \subseteq A$ and $B(x) \supseteq 0_\sim$.

Definition 3.7. An intuitionistic fuzzy set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ in a partially ordered set $(X,\ \leq)$ is said to be an

(i) increasing intuitionistic fuzzy set if $x \leq y$ implies $A(x) \subseteq A(y)$. That is, $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$.

(ii) decreasing intuitionistic fuzzy set if $x \leq y$ implies $A(x) \supseteq A(y)$. That is, $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$.

Definition 3.8. An ordered intuitionistic fuzzy bitopological space is an intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ (where τ_1 and τ_2 are intuitionistic fuzzy topologies on X) equipped with a partial order $\leq$.

Definition 3.9. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an upper pairwise intuitionistic fuzzy T_1-ordered space if $a,\ b \in X$ such that $a \nleq b$, there exists an decreasing τ_1 intuitionistic fuzzy neighbourhood or an decreasing τ_2 intuitionistic fuzzy neighbourhood A of b such that $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ is not an intuitionistic fuzzy neighbourhood of a.

Definition 3.10. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an lower pairwise intuitionistic fuzzy T_1-ordered space if $a,\ b \in X$ such that $a \nleq b$, there exists an increasing τ_1 intuitionistic fuzzy neighbourhood or an increasing τ_2 intuitionistic fuzzy neighbourhood A of a such that $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ is not an intuitionistic fuzzy neighbourhood of b.

Definition 3.11. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is

said to be an pairwise intuitionistic fuzzy T_1-ordered space if and only if it is both upper and lower pairwise intuitionistic fuzzy T_1-ordered space.

Definition 3.12. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an upper pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space if $a,\ b \in X$ such that $a \nleq b$, there exists an decreasing τ_1 intuitionistic fuzzy G_δ-α-locally neighbourhood or an decreasing τ_2 intuitionistic fuzzy G_δ-α-locally neighbourhood $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ of b such that A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of a.

Definition 3.13. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space if $a,\ b \in X$ such that $a \nleq b$, there exists an increasing τ_1 intuitionistic fuzzy G_δ-α-locally neighbourhood or an increasing τ_2 intuitionistic fuzzy G_δ-α-locally neighbourhood $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ of a such that A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b.

Definition 3.14. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space if and only if it is both upper and lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

Proposition 3.1. For an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ the following are equivalent:

(i) X is an lower (resp. upper) pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

(ii) For each $a,\ b \in X$ such that $a \nleq b$, there exists an increasing (resp. decreasing) τ_1 intuitionistic fuzzy G_δ-α-locally open set or an increasing (resp. decreasing) τ_2 intuitionistic fuzzy G_δ-α-locally open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ such that $A(a) > 0$ (resp. $A(b) > 0$) and A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b (resp. a).

Proof. (i)$\Rightarrow$(ii) Let X be an lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space. Let $a,\ b \in X$ such that $a \nleq b$. There exists an increasing τ_1 intuitionistic fuzzy G_δ-α-locally neighbourhood (or) an increasing τ_2 intuitionistic fuzzy G_δ-α-locally neighbourhood A of a such that A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b. It follows that there exists an τ_i intuitionistic fuzzy G_δ-α-locally open set ($i = 1$ or 2), $A_i = \langle x,\ \mu_{A_i},\ \gamma_{A_i}\rangle$ with $A_i \subseteq A$ and $A_i(a) = A(a) > 0$. As A is an increasing intuitionistic fuzzy set, $A(a) > A(b)$ and since A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b, $A_i(b) < A(b)$ implies $A_i(a) = A(a) > A(b) \geq A_i(b)$. This shows that A_i is an increasing intuitionistic fuzzy set and A_i is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b, since A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b.

(ii)$\Rightarrow$(i) Since A_1 is an increasing τ_1 intuitionistic fuzzy G_δ-α-locally open set or increasing τ_2 intuitionistic fuzzy G_δ-α-locally open set. Now, A_1 is an intuitionistic fuzzy G_δ-α-locally neighbourhood of a with $A_1(a) > 0$. By (ii), A_1 is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b. This implies, X is an lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

Remark 3.1. Similar proof holds for upper pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

Proposition 3.2. If $(X,\ \tau_1,\ \tau_2,\ \leq)$ is an lower (resp. upper) pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space and $\tau_1 \subseteq \tau_1^*,\ \tau_2 \subseteq \tau_2^*$, then $(X,\ {\tau_1}^*,\ {\tau_2}^*,\ \leq)$ is an lower (resp. upper) pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

Proof. Let $(X,\ \tau_1,\ \tau_2,\ \leq)$ be an lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space. Then if $a,\ b \in X$ such that $a \nleq b$, there exists an increasing τ_1 intuitionistic fuzzy G_δ-α-locally neighbourhood or an increasing τ_2 intuitionistic fuzzy G_δ-α-locally neighbourhood $A = \langle x,\ \mu_A,\ \gamma_A \rangle$ of a such that A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b. Since $\tau_1 \subseteq \tau_1^*$ and $\tau_2 \subseteq \tau_2^*$. Therefore, if $a,\ b \in X$ such that $a \nleq b$, there exists an increasing ${\tau_1}^*$ intuitionistic fuzzy G_δ-α-locally neighbourhood or an increasing ${\tau_2}^*$ intuitionistic fuzzy G_δ-α-locally neighbourhood $A = \langle x,\ \mu_A,\ \gamma_A \rangle$ of a such that A is not an intuitionistic fuzzy G_δ-α-locally neighbourhood of b. Thus $(X,\ {\tau_1}^*,\ {\tau_2}^*,\ \leq)$ is an lower pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

Remark 3.2. Similar proof holds for upper pairwise intuitionistic fuzzy G_δ-α-locally T_1-ordered space.

Definition 3.15. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an pairwise intuitionistic fuzzy T_2-ordered space if for $a,\ b \in X$ with $a \nleq b$, there exist an intuitionistic fuzzy open sets $A = \langle x,\ \mu_A,\ \gamma_A \rangle$ and $B = \langle x,\ \mu_B,\ \gamma_B \rangle$ such that A is an increasing τ_i intuitionistic fuzzy neighbourhood of a, B is an decreasing τ_j intuitionistic fuzzy neighbourhood of b $(i,\ j = 1,\ 2$ and $i \neq j)$ and $A \cap B = 0_\sim$.

Definition 3.16. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be an pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space if for $a,\ b \in X$ with $a \nleq b$, there exist an intuitionistic fuzzy G_δ-α-locally open sets $A = \langle x,\ \mu_A,\ \gamma_A \rangle$ and $B = \langle x,\ \mu_B,\ \gamma_B \rangle$ such that A is an increasing τ_i intuitionistic fuzzy G_δ-α-locally neighbourhood of a, B is an decreasing τ_j intuitionistic fuzzy G_δ-α-locally neighbourhood of b $(i,\ j = 1,\ 2$ and $i \neq j)$ and $A \cap B = 0_\sim$.

Definition 3.17. Let $(X,\ \leq)$ be a partially ordered set. Let $G = \{(x,\ y) \in X \times X \mid x \leq y,\ y = f(x)\}$. Then G is called an intuitionistic fuzzy graph of the partially ordered $\leq$.

Definition 3.18. Let $(X,\ T)$ be an intuitionistic fuzzy topological space and $A \subset X$ be a subset of X. An intuitionistic fuzzy characteristic function of $A = \langle x,\ \mu_A,\ \gamma_A \rangle$ is defined as

$$\chi_A(x) = \begin{cases} 1_\sim, & \text{if } x \in A, \\ 0_\sim, & \text{if } x \notin A. \end{cases}$$

Definition 3.19. Let $A = \langle x,\ \mu_A,\ \gamma_A \rangle$ be an intuitionistic fuzzy set in an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$. Then for $i = 1$ or 2, we define

$$\begin{aligned} I_{\tau_i} - G_\delta - \alpha - li(A) &= \text{increasing } \tau_i \text{ intuitionistic fuzzy } G_\delta - \alpha - \text{locally interior of } A \\ &= \text{the greatest increasing } \tau_i \text{ intuitionistic fuzzy } \mathrm{G}_\delta - \alpha - \text{locally} \\ &\quad\ \text{open set contained in } A. \end{aligned}$$

$$\begin{aligned} D_{\tau_i} - G_\delta - \alpha - li(A) &= \text{decreasing } \tau_i \text{ intuitionistic fuzzy } G_\delta - \alpha - \text{locally interior of } A \\ &= \text{the greatest decreasing } \tau_i \text{ intuitionistic fuzzy } \mathrm{G}_\delta - \alpha - \text{locally} \\ &\quad\ \text{open set contained in } A. \end{aligned}$$

$$
\begin{aligned}
I_{\tau_i} - G_\delta - \alpha - lc(A) &= \text{increasing } \tau_i \text{ intuitionistic fuzzy } G_\delta - \alpha - \text{locally closure of } A \\
&= \text{the smallest increasing } \tau_i \text{ intuitionistic fuzzy } \mathrm{G}_\delta - \alpha - \text{locally} \\
&\quad\ \text{closed set containing in } A.
\end{aligned}
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\begin{aligned}
D_{\tau_i} - G_\delta - \alpha - lc(A) &= \text{decreasing } \tau_i \text{ intuitionistic fuzzy } G_\delta - \alpha - \text{locally closure of } A \\
&= \text{the smallest decreasing } \tau_i \text{ intuitionistic fuzzy } \mathrm{G}_\delta - \alpha - \text{locally} \\
&\quad\ \text{closed set containing in } A.
\end{aligned}
$$

Notation 3.1. (i) The complement of the characteristic function χ_G, where G is the intuitionistic fuzzy graph of the partial order of X is denoted by $\chi_{\overline{G}}$.

(ii) I_{τ_i}-G_δ-α-$lc(A)$ is denoted by $I_i(A)$ and D_{τ_j}-G_δ-α-$lc(A)$ is denoted by $D_j(A)$, where $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ is an intuitionistic fuzzy set in an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$, for $i,\ j = 1,\ 2$ and $i \neq j$.

(iii) I_{τ_i}-G_δ-α-$li(A)$ is denoted by $I_i{}^\circ(A)$ and D_{τ_j}-G_δ-α-$li(A)$ is denoted by $D_j{}^\circ(A)$, where $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ is an intuitionistic fuzzy set in an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$, for $i,\ j = 1,\ 2$ and $i \neq j$.

Proposition 3.3. For an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ the following are equivalent:

(i) X is a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space.

(ii) For each pair $a,\ b \in X$ such that $a \nleq b$, there exist an τ_i intuitionistic fuzzy G_δ-α-locally open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ and τ_j intuitionistic fuzzy G_δ-α-locally open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ such that $A(a) > 0,\ B(b) > 0$ and $A(x) > 0,\ B(y) > 0$ together imply that $x \nleq y$.

(iii) The characteristic function χ_G, where G is the intuitionistic fuzzy graph of the partial order of X is a τ^*-intuitionistic fuzzy G_δ-α-locally closed set, where τ^* is either $\tau_1 \times \tau_2$ or $\tau_2 \times \tau_1$ in $X \times X$.

Proof. (i)$\Rightarrow$(ii) Let X be a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. Assume that suppose $A(x) > 0$, $B(y) > 0$ and $x \leq y$. Since A is an increasing τ_i intuitionistic fuzzy G_δ-α-locally open set and B is an decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set, $A(x) \leq A(y)$ and $B(y) \leq B(x)$. Therefore $0 < A(x) \cap B(y) \leq A(y) \cap B(x)$, which is a contradiction to the fact that $A \cap B = 0_\sim$. Therefore $x \nleq y$.

(ii)$\Rightarrow$(i) Let $a,\ b \in X$ with $a \nleq b$, there exists an intuitionistic fuzzy sets A and B satisfying the properties in (ii). Since $I_i{}^\circ(A)$ is an increasing τ_i intuitionistic fuzzy G_δ-α-locally open set and $D_j{}^\circ(B)$ is an decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set, we have $I_i{}^\circ(A) \cap D_j{}^\circ(B)$=$0_\sim$. Suppose $z \in X$ is such that $I_i{}^\circ(A)(z) \cap D_j{}^\circ(B)(z)$ >0. Then $I_i{}^\circ(A)$ >0 and $D_j{}^\circ(B)(z)$ >0. If $x \leq z \leq y$, then $x \leq z$ implies that $D_j{}^\circ(B)(x) \geq D_j{}^\circ(B)(z)$ >0 and $z \leq y$ implies that $I_i{}^\circ(A)(y) \geq I_i{}^\circ(A)(z)$ >0 then $D_j{}^\circ(B)(x)$ >0 and $I_i{}^\circ(A)(y)$ >0. Hence by (ii), $x \nleq y$ but then $x \leq y$. This is a contradiction. This implies that X is pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space.

(i)$\Rightarrow$(iii) We want to show that χ_G is an τ^* intuitionistionic fuzzy G_δ-α-locally closed set. That is to show that $\chi_{\overline{G}}$ is an τ^* intuitionistionic fuzzy G_δ-α-locally open set. It is sufficient to prove that $\chi_{\overline{G}}$ is an intuitionistionic fuzzy G_δ-α-locally neighbourhood of a point $(x,\ y) \in X \times X$ such that $\chi_{\overline{G}}\ (x,\ y) > 0$. Suppose $(x,\ y) \in X \times X$ is such that $\chi_{\overline{G}}\ (x,\ y) > 0$.

That is $\chi_G(x,\ y) < 1$. This means $\chi_G(x,\ y) = 0$. That is $(x,\ y) \notin G$. That is $x \nleq y$. Therefore by assumption (i), there exist intuitionistic fuzzy G_δ-α-locally open sets A and B such that A is an increasing τ_i intuitionistic fuzzy G_δ-α-locally neighbourhood of a, B is an decreasing τ_j intuitionistic fuzzy G_δ-α-locally neighbourhood of b ($i,\ j = 1,\ 2$ and $i \neq j$) and $A \cap B = 0_\sim$. Clearly $A \times B$ is an $IF\tau^*$ G_δ-α-locally neighbourhood of $(x,\ y)$. It is easy to verify that $A \times B \subseteq \chi_{\overline{G}}$. Thus we find that $\chi_{\overline{G}}$ is an τ^* IFG_δ-α-locally open set. Hence (iii) is established.

(iii)⇒(i) Suppose $x \nleq y$. Then $(x,\ y) \notin G$, where G is an intuitionistic fuzzy graph of the partial order. Given that χ_G is an τ^* intuitionistic fuzzy G_δ-α-locally closed set. That is $\chi_{\overline{G}}$ is an τ^* intuitionistic fuzzy G_δ-α-locally open set. Now $(x,\ y) \notin G$ implies that $\chi_{\overline{G}}(x,\ y) > 0$. Therefore $\chi_{\overline{G}}$ is an τ^* intuitionistic fuzzy G_δ-α-locally neighbourhood of $(x,\ y) \in X \times X$. Hence we can find that τ^* intuitionistic fuzzy G_δ-α-locally open set $A \times B$ such that $A \times B \subseteq \chi_{\overline{G}}$ and A is an τ_i intuitionistic fuzzy G_δ-α-locally open set such that $A(x) > 0$ and B is an τ_j intuitionistic fuzzy G_δ-α-locally open set such that $B(y) > 0$. We now claim that ${I_i}^\circ(A) \cap {D_j}^\circ(B)=0_\sim$. For if $z \in X$ is such that $({I_i}^\circ(A) \cap {D_j}^\circ(B))(z)> 0$, then ${I_i}^\circ(A)(z) \cap {D_j}^\circ(B)(z) > 0$. This means ${I_i}^\circ(A)(z)> 0$ and ${D_j}^\circ(B)(z)> 0$. And if $a \leq z \leq b$, then $z \leq b$ implies that ${I_i}^\circ(A)(b)\geq {I_i}^\circ(A)(z)> 0$ and $a \leq z$ implies that ${D_j}^\circ(B)(a)\geq {D_j}^\circ(B)(z)> 0$. Then ${D_j}^\circ(B)(a)> 0$ and ${I_i}^\circ(A)(b)> 0$ implies that $a \nleq b$ but then $a \leq b$. This is a contradiction. Hence (i) is established.

Definition 3.20. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be a weakly pairwise intuitionistic fuzzy T_2-ordered space if given $b < a$ (that is $b \leq a$ and $b \neq a$), there exist an τ_i intuitionistic fuzzy open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ such that $A(a) > 0$ and τ_j intuitionistic fuzzy open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ such that $B(b) > 0$ ($i,\ j = 1,\ 2$ and $i \neq j$) such that if $x,\ y \in X$, $A(x) > 0$, $B(y) > 0$ together imply that $y < x$.

Definition 3.21. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be a weakly pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space if given $b < a$ (that is $b \leq a$ and $b \neq a$), there exist an τ_i intuitionistic fuzzy G_δ-α-locally open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ such that $A(a) > 0$ and τ_j intuitionistic fuzzy G_δ-α-locally open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ such that $B(b) > 0$ ($i,\ j = 1,\ 2$ and $i \neq j$) such that if $x,\ y \in X$, $A(x) > 0$, $B(y) > 0$ together imply that $y < x$.

Definition 3.22. The symbol $x \parallel y$ means that $x \leq y$ and $y \leq x$.

Definition 3.23. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be a almost pairwise intuitionistic fuzzy T_2-ordered space if given $a \parallel b$, there exist an τ_i intuitionistic fuzzy open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ such that $A(a) > 0$ and τ_j intuitionistic fuzzy open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ such that $B(b) > 0$ ($i,\ j = 1,\ 2$ and $i \neq j$) such that if $x,\ y \in X$, $A(x) > 0$ and $B(y) > 0$ together imply that $x \parallel y$.

Definition 3.24. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be a almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space if given $a \parallel b$, there exist an τ_i intuitionistic fuzzy G_δ-α-locally open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ such that $A(a) > 0$ and τ_j intuitionistic fuzzy G_δ-α-locally open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ such that $B(b) > 0$ ($i,\ j = 1,\ 2$ and $i \neq j$) such that if $x,\ y \in X$, $A(x) > 0$ and $B(y) > 0$ together imply that $x \parallel y$.

Proposition 3.4. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space if and only if it is a weakly pairwise

intuitionistic fuzzy G_δ-α-locally T_2-ordered and almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space.

Proof. Let $(X,\ \tau_1,\ \tau_2,\ \leq)$ be a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. Then by Proposition 3.3 and Definition 3.20 it is a weakly pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. Let $a \parallel b$. Then $a \nleq b$ and $b \nleq a$. Since $a \nleq b$ and X is a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. We have τ_i intuitionistic fuzzy G_δ-α-locally open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ and τ_j intuitionistic fuzzy G_δ-α-locally open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ such that $A(a) > 0,\ B(b) > 0$ and $A(x) > 0,\ B(y) > 0$ together imply that $x \nleq y$. Also since $b \nleq a$, there exist τ_i intuitionistic fuzzy G_δ-α-locally open set $A^*{=}\langle x,\ \mu_{A^*},\ \gamma_{A^*}\rangle$ and τ_j intuitionistic fuzzy G_δ-α-locally open set $B^*{=}\langle x,\ \mu_{B^*},\ \gamma_{B^*}\rangle$ such that $A^*(a) > 0,\ B^*(b) > 0$ and $A^*(x) > 0,\ B^*(y) > 0$ together imply that $y \nleq x$. Thus $I_i{}^\circ(A \cap A^*)$ is an τ_i intuitionistic fuzzy G_δ-α-locally open set such that $I_i{}^\circ(A\cap A^*)(a) > 0$ and $I_j{}^\circ(B\cap B^*)$ is an τ_j intuitionistic fuzzy G_δ-α-locally open set such that $I_j{}^\circ(B \cap B^*)(b) > 0$. Also $I_i{}^\circ(A \cap A^*)(x) > 0$ and $I_j{}^\circ(B\cap B^*)(y) > 0$ togetherimply that $x \parallel y$. Hence X is a almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space.

Conservely, let X be a weakly pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered and almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. We want to show that X is a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. Let $a \nleq b$. Then either $b < a$ or $b \nleq a$. If $b < a$ then X being weakly pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space, there exist τ_i intuitionistic fuzzy G_δ-α-locally open set A and τ_j intuitionistic fuzzy G_δ-α-locally open set B such that $A(a) > 0,\ B(b) > 0$ and such that $A(x) > 0,\ B(y) > 0$ together imply that $y < x$. That is $x \nleq y$. If $b \nleq a$, then $a \parallel b$ and the result follows easily since X is a almost pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space. Hence X is a pairwise intuitionistic fuzzy G_δ-α-locally T_2-ordered space.

Definition 3.25. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ and $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ be intuitionistic fuzzy sets in an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$. Then A is said to be an τ_i intuitionistic fuzzy neighbourhood of B if $B \subseteq A$ and there exists τ_i intuitionistic fuzzy open set $C = \langle x,\ \mu_C,\ \gamma_C\rangle$ such that $B \subseteq C \subseteq A$ $(i = 1$ or $2)$.

Definition 3.26. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ and $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ be intuitionistic fuzzy sets in an ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$. Then A is said to be an τ_i intuitionistic fuzzy G_δ-α-locally neighbourhood of B if $B \subseteq A$ and there exists τ_i intuitionistic fuzzy G_δ-α-locally open set $C = \langle x,\ \mu_C,\ \gamma_C\rangle$ such that $B \subseteq C \subseteq A$ $(i = 1$ or $2)$.

Definition 3.27. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is said to be a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space if for every pair $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ is an decreasing τ_i intuitionistic fuzzy G_δ-α-locally closed set and $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ is an decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set such that $A \subseteq B$ then there exist decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set $A_1{=}\langle x,\ \mu_{A_1},\ \gamma_{A_1}\rangle$ such that $A \subseteq A_1 \subseteq D_i(A_1) \subseteq B$ $(i,\ j = 1,\ 2$ and $i \neq j)$.

Proposition 3.5. An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ the following are equivalent:

(i) $(X,\ \tau_1,\ \tau_2,\ \leq)$ is a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space.

(ii) For each increasing τ_i intuitionistic fuzzy G_δ-α-locally open set $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ and decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ with $A \subseteq B$ there exists an decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set A_1 such that $A \subseteq A_1 \subseteq IFG_\delta$-$\alpha$-$lcl_{\tau_i}(A_1) \subseteq B$ $(i,\ j = 1,\ 2$ and $i \neq j)$.

Proof. The proof is simple.

Definition 3.28. Let $(X,\ \tau_1,\ \tau_2,\ \leq)$ be an ordered intuitionistic fuzzy bitopological space. A function $f:\ X \to R(I)$ is said to be an τ_i *lower**(resp. *upper**) intuitionistic fuzzy G_δ-α-locally continuous function if $f^{-1}(R_t)$ (resp. $f^{-1}(L_t)$) is an increasing or an decreasing τ_i (resp. τ_j) intuitionistic fuzzy G_δ-α-locally open set, for each $t \in R$ $(i,\ j = 1,\ 2$ and $i \neq j)$.

Proposition 3.6. Let $(X,\ \tau_1,\ \tau_2,\ \leq)$ be an ordered intuitionistic fuzzy bitopological space. Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ be an intuitionistic fuzzy set in X and let $f:\ X \to R(I)$ be such that

$$f(x)(t) = \begin{cases} 1, & \text{if} \quad t < 0, \\ A(x), & \text{if} \quad 0 \leq t \leq 1, \\ 0, & \text{if} \quad t > 1. \end{cases}$$

for all $x \in X$. Then f is an τ_i *lower**(resp. τ_j *upper**) intuitionistic fuzzy G_δ-α-locally continuous function if and only if A is an increasing or an decreasing τ_i (resp. τ_j) intuitionistic fuzzy G_δ-α-locally open (resp. closed) set $(i,\ j = 1,\ 2$ and $i \neq j)$.

Proof.

$$f^{-1}(R_t) = \begin{cases} 1, & \text{if} \quad t < 0, \\ A(x), & \text{if} \quad 0 \leq t \leq 1, \\ 0, & \text{if} \quad t > 1. \end{cases}$$

implies that f is an τ_i *lower** intuitionistic fuzzy G_δ-α-locally continuous function if and only if A is an increasing or an decreasing τ_i intuitionistic fuzzy G_δ-α-locally open set in X.

$$f^{-1}(L_t) = \begin{cases} 1, & \text{if} \quad t < 0, \\ A(x), & \text{if} \quad 0 \leq t \leq 1, \\ 0, & \text{if} \quad t > 1. \end{cases}$$

implies that f is an τ_j *upper** intuitionistic fuzzy G_δ-α-locally continuous function if and only if A is an increasing or an decreasing τ_j intuitionistic fuzzy G_δ-α-locally closed set in X $(i,\ j = 1,\ 2$ and $i \neq j)$.

Proposition 3.7. (Uryshon's lemma) An ordered intuitionistic fuzzy bitopological space $(X,\ \tau_1,\ \tau_2,\ \leq)$ is a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space if and only if for every $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ is an decreasing τ_i intuitionistic fuzzy closed set and $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ is an increasing τ_j intuitionistic fuzzy closed set with $A \subseteq \overline{B}$, there exists increasing intuitionistic fuzzy function $f:\ X \to I$ such that $A \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq B$ and f is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function $(i,\ j = 1,\ 2$ and $i \neq j)$.

Proof. Suppose that there exists a function f satisfying the given conditions. Let $C = \langle x,\ \mu_C,\ \gamma_C\rangle = f^{-1}(\overline{L_t})$ and $D = \langle x,\ \mu_D,\ \gamma_D\rangle = f^{-1}(R_t)$ for some $0 \leq t \leq 1$. Then $\overline{C} \in \tau_i$ and $D \in \tau_j$ and such that $A \subseteq C \subseteq D \subseteq \overline{B}$. It is easy to verify that D is an decreasing

τ_j intuitionistic fuzzy G_δ-α-locally open set and C is an increasing τ_i intuitionistic fuzzy G_δ-α-locally closed set. Then there exists decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set C_1 such that $C \subseteq C_1 \subseteq D_i(C_1) \subseteq D$ $(i,\ j = 1,\ 2$ and $i \neq j)$. This proves that X is a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space. Conversely, let X be a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space. Let A be an decreasing τ_i intuitionistic fuzzy G_δ-α-locally closed set and B be an increasing τ_j intuitionistic fuzzy G_δ-α-locally closed set. By the Proposition 3.6, we can construct a collection $\{C_t \mid t \in I\} \subseteq \tau_j$, where $C = \langle x,\ \mu_{C_t},\ \gamma_{C_t}\rangle$, $t \in I$ such that $A \subseteq C_t \subseteq B$, $IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(C_s) \subseteq C_t$ whenever $s < t$, $A \subseteq C_0$ $C_1 = B$ and $C_t = 0_\sim$ for $t < 0$, $C_t = 1_\sim$ for $t > 1$. We define a function $f:\ X \to I$ by $f(x)(t) = C_{1-t}(x)$. Clearly f is well defined. Since $A \subseteq C_{1-t} \subseteq B$, for $t \in I$. We have $A \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq B$. Furthermore $f^{-1}(R_t) = \bigcup_{s<1-t} C_s$ is an τ_j intuitionistic fuzzy G_δ-α-locally open set and $f^{-1}(\overline{L_t}) = \bigcap_{s>1-t} C_s = \bigcap_{s>1-t} IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(C_s)$ is an τ_i intuitionistic fuzzy G_δ-α-locally closed set. Thus f is an τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function and τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and is an increasing intuitionistic fuzzy function.

Proposition 3.8. (Tietze extension theorem) Let $(X,\ \tau_1,\ \tau_2,\ \leq)$ be an ordered intuitionistic fuzzy bitopological space the following statements are equivalent:

(i) $(X,\ \tau_1,\ \tau_2,\ \leq)$ is a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space.

(ii) If $g,\ h:\ X \to R(I)$, g is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function, h is an τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function and $g \subseteq h$, then there exists $f:\ X \to R(I)$ such that $g \subseteq f \subseteq h$ and f is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function$(i,\ j = 1,\ 2$ and $i \neq j)$.

Proof. (ii)$\Rightarrow$(i) Let $A = \langle x,\ \mu_A,\ \gamma_A\rangle$ and $B = \langle x,\ \mu_B,\ \gamma_B\rangle$ be an intuitionistic fuzzy G_δ-α-locally open sets such that $A \subseteq B$. Define $g,\ h:\ X \to R(I)$ by

$$g(x)(t) = \begin{cases} 1, & \text{if} \quad t < 0, \\ A(x), & \text{if} \quad 0 \leq t \leq 1, \\ 0, & \text{if} \quad t > 1. \end{cases}$$

and

$$h(x)(t) = \begin{cases} 1, & \text{if} \quad t < 0, \\ B(x), & \text{if} \quad 0 \leq t \leq 1, \\ 0, & \text{if} \quad t > 1. \end{cases}$$

for each $x \in X$. By Proposition 3.6, g is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and h is an τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function. Clearly, $g \subseteq h$ holds, so that there exists $f:\ X \to R(L)$ such that $g \subseteq f \subseteq h$. Suppose $t \in (0,\ 1)$. Then $A = g^{-1}(R_t) \subseteq f^{-1}(R_t) \subseteq f^{-1}(\overline{L_t}) \subseteq h^{-1}(\overline{L_t}) = B$. By Proposition 3.7, X is a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space.

(i)$\Rightarrow$(ii) Define two mappings $A,\ B:\ Q \to I$ by $A(r) = A_r = h^{-1}(\overline{R_r})$ and $B(r) = B_r = g^{-1}(L_r)$, for all $r \in Q$ (Q is the set of all rationals). Clearly, A and B are monotone increasing families of an decreasing τ_i intuitionistic fuzzy G_δ-α-locally closed sets and decreasing τ_j intuitionistic fuzzy G_δ-α-locally open sets of X. Moreover $A_r \subset B_{r'}$ if $r < r'$. By Proposition 3.5, there exists an decreasing τ_j intuitionistic fuzzy G_δ-α-locally open set $C = \langle x,\ \mu_C,\ \gamma_C\rangle$ such that $A_r \subseteq IFG_\delta\text{-}\alpha\text{-}lint_{\tau_i}(C_r)$, $IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(C_r) \subseteq IFG_\delta\text{-}\alpha\text{-}lint_{\tau_i}(C_{r'})$, $IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(C_r) \subseteq B_{r'}$ whenever $r < r'$ ($r,\ r' \in Q$). Letting $V_t = \bigcap_{r<t} \overline{C_r}$ for $t \in R$, we define a monotone decreasing family $\{V_t \mid t \in R\} \subseteq I$. Moreover we have $IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(V_t) \subseteq IFG_\delta\text{-}\alpha\text{-}lint_{\tau_i}(V_s)$ whenever $s < t$. We have,

$$\begin{aligned}\bigcup_{t\in R} V_t &= \bigcup_{t\in R}\bigcap_{r<t}\overline{C_r} \supseteq \bigcup_{t\in R}\bigcap_{r<t}\overline{B_r} = \bigcup_{t\in R}\bigcap_{r<t} g^{-1}(\overline{L_r}) = \bigcup_{t\in R} g^{-1}(\overline{L_t}) = g^{-1}(\bigcup_{t\in R}\overline{L_t})\\ &= 1_\sim.\end{aligned}$$

Similarly, $\bigcap_{t\in R} V_t = 0_\sim$. Now define a function $f:(X,\ \tau_1,\ \tau_2,\ \leq) \to R(L)$ satisfying the required conditions. Let $f(x)(t) = V_t(x)$, for all $x \in X$ and $t \in R$. By the above discussion, it follows that f is well defined. To prove f is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function ($i,\ j = 1,\ 2$ and $i \neq j$). Observe that $\bigcup_{s>t} V_s = \bigcup_{s>t} IFG_\delta\text{-}\alpha\text{-}lint_{\tau_i}(V_s)$ and $\bigcap_{s>t} V_s = \bigcap_{s>t} IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(V_s)$. Then $f^{-1}(R_t) = \bigcup_{s>t} V_s = \bigcup_{s>t} IFG_\delta\text{-}\alpha\text{-}lint_{\tau_i}(V_s)$ is an increasing τ_i intuitionistic fuzzy G_δ-α-locally open set. Now $f^{-1}(\overline{L_t}) = \bigcap_{s>t} V_s = \bigcap_{s>t} IFG_\delta\text{-}\alpha\text{-}lcl_{\tau_i}(V_s)$ is an decreasing τ_j intuitionistic fuzzy G_δ-α-locally closed set. So that f is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function. To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is $g^{-1}(\overline{L_t}) \subseteq f^{-1}(\overline{L_t}) \subseteq h^{-1}(\overline{L_t})$ and $g^{-1}(R_t) \subseteq f^{-1}(R_t) \subseteq h^{-1}(R_t)$ for each $t \in R$. We have,

$$g^{-1}(\overline{L_t}) = \bigcap_{s<t} g^{-1}(\overline{L_s}) = \bigcap_{s<t}\bigcap_{r<s} g^{-1}(\overline{L_r}) = \bigcap_{s<t}\bigcap_{r<s}\overline{B_r} \subseteq \bigcap_{s<t}\bigcap_{r<s}\overline{C_r} = \bigcap_{s<t} V_s = f^{-1}(\overline{L_t})$$

and

$$f^{-1}(\overline{L_t}) = \bigcap_{s<t} V_s = \bigcap_{s<t}\bigcap_{r<s}\overline{C_r} \subseteq \bigcap_{s<t}\bigcap_{r<s}\overline{A_r} = \bigcap_{s<t}\bigcap_{r<s} h^{-1}(\overline{R_r}) = \bigcap_{s<t} h^{-1}(\overline{L_s}) = h^{-1}(\overline{L_t}).$$

Similarly, we obtain

$$g^{-1}(R_t) = \bigcup_{s>t} g^{-1}(R_s) = \bigcup_{s>t}\bigcup_{r>s} g^{-1}(\overline{L_r}) = \bigcup_{s>t}\bigcup_{r>s}\overline{B_r} \subseteq \bigcup_{s>t}\bigcup_{r>s}\overline{C_r} = \bigcup_{s>t} V_s = f^{-1}(\overline{R_t})$$

and

$$f^{-1}(\overline{R_t}) = \bigcup_{s>t} V_s = \bigcup_{s>t}\bigcup_{r>s}\overline{C_r} \subseteq \bigcup_{s>t}\bigcup_{r>s}\overline{A_r} = \bigcup_{s>t}\bigcup_{r>s} h^{-1}(\overline{R_r}) = \bigcup_{s>t} h^{-1}(\overline{R_s}) = h^{-1}(\overline{R_t}).$$

Hence the proof.

Proposition 3.9. Let $(X,\ \tau_1,\ \tau_2,\ \leq)$ be a strongly pairwise intuitionistic fuzzy G_δ-α-locally normally ordered space. Let $\overline{A} \in \tau_1$ and $\overline{A} \in \tau_2$ be crisp and let $f:\ (A,\ \tau_1/A,\ \tau_2/A) \to I$ be an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function ($i,\ j = 1,\ 2$ and $i \neq j$). Then f has an intuitionistic fuzzy extension over $(X,\ \tau_1,\ \tau_2,\ \leq)$ (that is, $F:\ (X,\ \tau_1,\ \tau_2,\ \leq) \to I$).

Proof. Define $g:\ X \to I$ by

$$g(x) = f(x), \quad \text{if} \quad x \in A; \quad g(x) = [A_0], \quad \text{if} \quad x \notin A$$

and also define $h: X \to I$ by

$$h(x) = f(x), \quad \text{if} \quad x \in A; \quad h(x) = [A_1], \quad \text{if} \quad x \notin A.$$

where $[A_0]$ is the equivalence class determined by $A_0: R \to I$ such that

$$A_0(t) = 1_\sim, \quad \text{if} \quad t < 0; \quad A_0(t) = 0_\sim, \quad \text{if} \quad t > 0$$

and $[A_1]$ is the equivalence class determined by $A_1: R \to I$ such that

$$A_1(t) = 1_\sim, \quad \text{if} \quad t < 1; \quad A_1(t) = 0_\sim, \quad \text{if} \quad t > 1.$$

g is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and h is an τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function and $g \subseteq h$. Hence by Proposition 3.8, there exists a function $F: X \to I$ such that F is an τ_i *upper** intuitionistic fuzzy G_δ-α-locally continuous function and τ_j *lower** intuitionistic fuzzy G_δ-α-locally continuous function and $g(x) \subseteq h(x) \subseteq f(x)$ for all $x \in X$. Hence for all $x \in A$, $f(x) \subseteq F(x) \subseteq f(x)$. So that F is an required extension of f over X.

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The χ^2 sequence spaces defined by a modulus

N. Subramanian[†], U. K. Misra[‡] and Vladimir Rakocevic[♯]

† Department of Mathematics, SASTRA University,
Tanjore, 613402, India
‡ Department of Mathematics, Berhampur University,
Berhampur, 760007, Orissa, India
♯ Faculty of Mathematics and Sciences,University of Nis,
Visegradska 33, 18000 Nis Serbia
E-mail: nsmaths@yahoo.com umakanta_misra@yahoo.com vrakoc@bankerinter.net

Abstract In this paper will be accomplished by presenting the following sequence spaces $\left\{x \in \chi^2 : P\text{-}\lim_{k,\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(((m+n)!\ |x_{mn}|)^{\frac{1}{m+n}}\right) = 0\right\}$ and $\left\{x \in \Lambda^2 : \sup_{k,\ell} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} a_{k\ell}^{mn} f\left(|x_{mn}|^{\frac{1}{m+n}}\right) < \infty\right\}$, where f is a modulus function and A is a nonnegative four dimensional matrix. We shall established inclusion theorems between these spaces and also general properties are discussed.

Keywords Gai sequence, analytic sequence, modulus function, double sequences.
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§1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m,\ n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [8], Moricz [12], Moricz and Rhoades [13], Basarir and Solankan [2], Tripathy [20], Colak and Turkmenoglu [6], Turkmenoglu [22], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,\ n \in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p\text{-}\lim_{m,\ n \to \infty} |x_{mn} - l|^{t_{mn}} = 1 \ \text{ for some } l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p\text{-}\lim_{m,\ n \to \infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,\ n \in \mathbb{N}$ and p - $\lim_{m,\ n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,\ n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α-, β-, γ- duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi^2_M(p,\ q,\ u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [42], Maddox [43] and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [11] as an extension of the definition of strongly Cesàro summable sequences. Connor [44] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A- statistical convergence. In [45] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [46] – [49] and [50] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary (single) sequence spaces to multiply sequence spaces. This will be accomplished by presenting the following sequence spaces:

$$\left\{ x \in \chi^2 : P\text{-}\lim_{k,\ \ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(((m+n)!\ |x_{mn}|)^{\frac{1}{m+n}} \right) = 0 \right\}$$

and

$$\left\{x \in \Lambda^2 : \sup_{k,\ \ell} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} a_{k\ell}^{mn} f\left(|x_{mn}|^{\frac{1}{m+n}}\right) < \infty\right\},$$

where f is a modulus function and A is a nonnegative four dimensional matrix. Other implications, general properties and variations will also be presented.

We need the following inequality in the sequel of the paper. For $a,\ b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,\ n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\ |x_{mn}|)^{1/m+n} \to 0$ as $m,\ n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,\ n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m,\ n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,\ j)^{th}$ place for each $i,\ j \in \mathbb{N}$.

An FK-space or a metric space X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m,\ n \in \mathbb{N})$ are also continuous.

Orlicz [16] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [17], Mursaleen et al.[14], Bektas and Altin [3], Tripathy et al.[21], Rao and Subramanian [18], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [9].

Recalling [16] and [9], an Orlicz function is a function $M :\ [0,\ \infty) \to [0,\ \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0,\ M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [15] and further discussed by Ruckle [19] and Maddox [11] and many others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2-condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ \text{for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) $X' =$ the continuous dual of X,

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{for each } x \in X\right\}$,

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X\right\}$,

(iv) $X^\gamma = \left\{a = (a_{mn}) : \sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,\ N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$,

(v) Let X be an FK-space $\supset \phi$; then $X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$,

(vi) $X^\delta = \left\{a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\right\}$,

$X^\alpha,\ X^\beta,\ X^\gamma$ are called α- (or $K\ddot{o}the$-Toeplitz) dual of X, β- (or generalized-$K\ddot{o}the$-Toeplitz) dual of X, γ- dual of X, δ- dual of X respectively. X^α is defined by Gupta and Kamptan [24]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{x = (x_k) \in w :\ (\Delta x_k) \in Z\},$$

for $Z = c,\ c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here $w,\ c,\ c_0$ and ℓ_∞ denote the classes of all, convergent, null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|.$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 :\ (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2,\ \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m,\ n \in \mathbb{N}$.

§2. Definitions and preliminaries

Throughout the article w^2 denotes the spaces of all sequences. χ^2_M and Λ^2_M denote the Pringscheims sense of double Orlicz space of gai sequences and Pringscheims sense of double Orlicz space of bounded sequences respecctively.

Definition 2.1. A modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0,\ \infty) \to [0,\ \infty)$, such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0,\ y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition,

(v) that f is continuous on $[0, \infty)$.

Definition 2.2. Let $p,\ q$ be semi norms on a vector space X. Then p is said to be stronger that q if whenever (x_{mn}) is a sequence such that $p(x_{mn}) \to 0$, then also $q(x_{mn}) \to 0$. If each is stronger than the others, the p and q are said to be equivalent.

Lemma 2.1. Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$ for all $x \in X$.

Definition 2.3. A sequence space E is said to be solid or normal if $(\alpha_{mn} x_{mn}) \in E$ whenever $(x_{mn}) \in E$ and for all sequences of scalars (α_{mn}) with $|\alpha_{mn}| \leq 1$, for all $m,\ n \in \mathbb{N}$.

Definition 2.4. A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.1. From the two above definitions it is clear that a sequence space E is solid implies that E is monotone.

Definition 2.5. A sequence E is said to be convergence free if $(y_{mn}) \in E$ whenever $(x_{mn}) \in E$ and $x_{mn} = 0$ implies that $y_{mn} = 0$.

By the gai of a double sequence we mean the gai on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Pringsheim limit 0 (denoted by P-lim x=0) such that $((m+n)!\ |x_{mn}|)^{\frac{1}{m+n}} = 0$, whenever $m,\ n \in \mathbb{N}$. We shall denote the space of all P- gai sequences by χ^2. The double sequence x is analytic if there exists a positive number M such that $|x_{jk}|^{\frac{1}{j+k}} < M$ for all j and k. We will denote the set of all analytic double sequences by Λ^2.

Throughout this paper we shall examine our sequence spaces using the following type of transformation:

Defintion 2.6. Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ \ell$- th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn},$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toeplitz and [51] and [52] respectively. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in whcih they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded.

Definition2.7. The four dimensional matrix A is said to be RH-regular if it maps every bounded P-gai sequence into a P-gai sequnece with the same P-limit.

In addition to this definition, Robison and Hamilton also presented the following Silverman-Toeplitz type multidimensional characterization of regularity in [50] and [46]:

Theorem 2.1. The four dimensional matrix A is RH-regular if and only if

$RH_1:$ $P\text{-}\lim_{k,\ \ell} a_{k\ell}^{mn} = 0$ for each m and n,

$RH_2:$ $P\text{-}\lim_{k,\ \ell} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} = 1,$

$RH_3:$ $P\text{-}\lim_{k,\ \ell} \sum_{m=1}^{\infty} |a_{k\ell}^{mn}| = 0$ for each n,

$RH_4:$ $P\text{-}\lim_{k,\ \ell} \sum_{n=1}^{\infty} |a_{k\ell}^{mn}| = 0$ for each m,

$RH_5:$ $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn}$ is P-convergent and

$RH_6:$ there exist positive numbers A and B such that $\sum_{m,\ n>B} |a_{k\ell}^{mn}| < A$.

Definition 2.8. A double sequence (x_{mn}) of complex numbers is said to be strongly A-summable to 0, if $P\text{-}\lim_{k,\ \ell} \sum_{m,\ n} a_{k\ell}^{mn} \left((m+n)!\ |x_{mn} - 0|\right)^{\frac{1}{m+n}} = 0$.

Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)),\ m = 1,\ 2,\ 3,\ \cdots$. A continuous linear functional ϕ on Λ^2 is said to be an invariant mean or a σ-mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_{mn})$ has $x_{mn} \geq 0$ for all $m,\ n$.

(ii) $\phi(e) = 1$ where

$$e = \begin{pmatrix} 1, & 1, & ... & 1 \\ 1, & 1, & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1, & 1, & ... & 1 \end{pmatrix},$$

(iii) $\phi(\{x_{\sigma(m),\sigma(n)}\}) = \phi(\{x_{mn}\})$ for all $x \in \Lambda^2$.

For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space C of all real convergent sequences in the sense that $\phi(x)$=lim x for all $x \in C$ consequently $C \subset V_\sigma$, where V_σ is the set of double analytic sequences all of those σ- means are equal.

If $x = (x_{mn})$, set $Tx = (Tx)^{1/m+n} = (x_{\sigma(m),\ \sigma(n)})$. It can be shown that

$$V_\sigma = \left\{x \in \Lambda^2 : \underset{m\to\infty}{im}\, t_{mn}(x_n)^{1/n} = \text{Le uniformly in } n,\ L = \sigma\text{-lim}(x_{mn})^{1/m+n}\right\},$$

where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{1/m+n}}{m+1}. \tag{2}$$

We say that a double analytic sequence $x = (x_{mn})$ is σ- convergent if and only if $x \in V_\sigma$.

Definition 2.9. A double analytic sequence $x = (x_{mn})$ of real numbers it said to be σ-convergent to zero provided that

$$P\text{-}\lim_{p,\ q} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} \left|x_{\sigma^m(k),\ \sigma^m(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^m(\ell)}} = 0, \text{ uniformly in } (k,\ \ell).$$

In this case we write σ_2- $\lim x$=0. We shall also denoted the set of all double σ-convergent sequences by V_σ^2. Clearly $V_\sigma^2 \subset \Lambda^2$.

One can see that in contrast to the case for single sequences, a P-convergent double sequence need not be σ-convergent. But, it is easy to see that every bounded P-convergent double sequence is convergent. In addition, if we let $\sigma(m) = m + 1$ and $\sigma(n) = n + 1$ in then σ-convergence of double sequences reduces to the almost convergence of double sequences.

The following definition is a combination of strongly A-summable to zero, modulus function and σ-convergent.

Definition2.10. Let f be a modulus, $A = (a_{k\ell}^{mn})$ be a nonnegative RH-regular summability matrix method and

$$e = \begin{pmatrix} 1, & 1, & ... & 1 \\ 1, & 1, & ... & 1 \\ . & & & \\ . & & & \\ . & & & \\ 1, & 1, & ... & 1 \end{pmatrix}.$$

We now define the following sequence spaces:

$$\begin{aligned} & \chi^2(A,\ f) \\ = & \left\{ x \in \chi^2 : P\text{-}\lim_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left((\sigma^m(k) + \sigma^n(\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} = 0 \right\}, \\ & \Lambda^2(A,\ f) \\ = & \left\{ x \in \Lambda^2 : \sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty \right\}. \end{aligned}$$

If $f(x) = x$ then the sequence spaces defined above reduce to the following:

$$\begin{aligned} & \chi^2(A) \\ = & \left\{ x \in \chi^2 : P\text{-}\lim_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left((\sigma^m(k) + \sigma^n(\ell))!\ \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} = 0 \right\} \end{aligned}$$

and

$$\begin{aligned} & \Lambda^2(A) \\ = & \left\{ x \in \Lambda^2 : \sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left(\left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty \right\}. \end{aligned}$$

Some well-known spaces are defined by specializing A and f. For example, if $A = (C,\ 1,\ 1)$ the

sequence spaces defined above reduces to $\chi^2(f)$ and $\Lambda^2(f)$ respectively

$$\begin{aligned}&\chi^2(f)\\=&\left\{x\in\chi^2: P\text{-}\lim_{k\ell}\frac{1}{k\ell}\sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1} f\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}=0\right\},\\&\Lambda^2(f)\\=&\left\{x\in\Lambda^2: \sup_{k\ell}\frac{1}{k\ell}\sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}<\infty\right\}.\end{aligned}$$

As a final illustration, let $A=(C,\ 1,\ 1)$ and $f(x)=x$, we obtainthe following spaces:

$$\chi^2=\left\{x\in\chi^2: P\text{-}\lim_{k\ell}\frac{1}{k\ell}\sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1}\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}=0\right\}$$

and

$$\Lambda^2=\left\{x\in\Lambda^2: \sup_{k\ell}\frac{1}{k\ell}\sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1}\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}<\infty\right\}.$$

§3. Main results

In this section we shall establish some general properties for the above sequence spaces.

Theorem 3.1. $\chi^2(A,\ f)$ and $\Lambda^2(A,\ f)$ are linear spaces over the complex filed $\mathbb{C}$.

Proof. We shall establish the linearity $\chi^2(A,\ f)$ only. The other cases can be treated in a similar manner. Let x and y be elements in $\chi^2(A,\ f)$. For λ and μ in $\mathbb{C}$ there exist integers M_λ and N_μ such tht $|\lambda|<M_\lambda$ and $|\mu|<N_\mu$. From the conditions (ii) and (iii) of Definition 2.1, we granted the following

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|\lambda x_{\sigma^m(k),\ \sigma^n(\ell)}+\mu y_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\\ \leq\ & M_\lambda\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\\&+N_\mu\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|y_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}},\end{aligned}$$

for all k and ℓ. Since x and y are $\chi^2(A,\ f)$, we have $\lambda x+\mu y\in\chi^2(A,\ f)$. Thus $\chi^2(A,f)$ is a linear space. This completes the proof.

Theorem 3.2. $\chi^2(A,\ f)$ is a complete linear topological spaces with the the paranorm

$$g(x)=\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}.$$

Proof. For each $x\in\chi^2(A,\ f)$, $g(x)$ is exists. Clearly $g(\theta)=0,\ g(-x)=g(x)$, and $g(x+y)\leq g(x)+g(y)$. We now show that the scalar multiplication is continuous. Now observe the following:

$$g(\lambda x)=\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left(\left|\lambda x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\leq(1+[\lambda])\, g(x).$$

Where $\left[|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}}\right]$ denotes the integer part of $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}}$. In addition observe that $g(x)$ and λ approahces 0 implies $g(\lambda x)$ approaches 0. For fixed λ, if x approaches 0 then $g(\lambda x)$ approaches 0. We now show that for fixed x, λ approaches 0 implies $g(\lambda x)$ approaches 0. Let $x \in \chi^2(A,\ f)$, thus

$P\text{-}\lim_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) = 0$.

If $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}} < 1$ and $M \in \mathbb{N}$ we have:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \lambda \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \\
\le\ & \sum_{m\le M}^{\infty}\sum_{n\le M}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \lambda \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \\
&+ \sum_{m\ge M}^{\infty}\sum_{n\ge M}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \lambda \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right).
\end{aligned}$$

Let $\epsilon > 0$ and choose N such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) < \frac{\epsilon}{2}, \tag{3}$$

for $k,\ \ell > N$. Also for each $(k,\ \ell)$ with $1 \le k \le N, 1 \le \ell \le N$, and since

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) < \infty,$$

there exists an integer $M_{k,\ell}$ such that

$$\sum_{m>M_{k,\ell}}\sum_{n>M_{k,\ell}} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) < \frac{\epsilon}{2}.$$

Taking $M = \inf_{1\le k\le N(or)1\le \ell\le N}\{M_{k,\ell}\}$, we have for each $(k,\ \ell)$ with $1 \le k \le \mathbb{N}$ or $1 \le \ell \le \mathbb{N}$.

$$\sum_{m>M}\sum_{n>M} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) < \frac{\epsilon}{2}.$$

Also from (3), for $k,\ \ell > N$ we have

$$\sum_{m>M}\sum_{n>M} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) < \frac{\epsilon}{2}.$$

Thus M is an integer independent of $(k,\ \ell)$ such that

$$\sum_{m>M}\sum_{n>M} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) < \frac{\epsilon}{2}. \tag{4}$$

Further for $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}} < 1$ and for all $(k,\ \ell)$,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| \lambda x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \\
\le\ & \sum_{m>M}\sum_{n>M} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| \lambda x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \\
&+ \sum_{m\le M}\sum_{n\le M} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left| \lambda x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right).
\end{aligned}$$

For each $(k,\ \ell)$ and by the continuity of f as $\lambda \to 0$ we have the following:

$$\sum_{m \leq M} \sum_{n \leq M} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|\lambda x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right).$$

Now a choose $\delta < 1$ such that $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}} < \delta$ implies

$$\sum_{m \leq M} \sum_{n \leq M} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|\lambda x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2}. \quad (5)$$

It follows that $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|\lambda x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon$ for all $(k,\ \ell)$. Thus $g(\lambda x) \to 0$ as $\lambda \to 0$. Therefore $\chi^2(A,\ f)$ is a paranormed linear topological space.

Now let us show that $\chi^2(A,\ f)$ is complete with respect to its paranorm topologies. Let $\left(x_{mn}^s\right)$ be a cauchy sequence in $\chi^2(A,\ f)$. Then, we write $g\left(x^s - x^t\right) \to 0$ as $s,\ t \to \infty$, to mean, as $s,\ t \to \infty$ for all $(k,\ \ell)$,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}^s - x_{\sigma^m(k),\ \sigma^n(\ell)}^t\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \to 0. \quad (6)$$

Thus for each fixed m and n as $s,\ t \to \infty$. We are granted

$$f\left((m+n)! \ \left|x_{mn}^s - x_{mn}^t\right|\right) \to 0$$

and so $\left(x_{mn}^s\right)$ is a cauchy sequence in $\mathbb{C}$ for each fixed m and n. Since $\mathbb{C}$ is complete as $s \to \infty$ we have $x_{mn}^s \to x_{mn}$ for each (mn). Now from Definition 2.9, we have for $\epsilon > 0$ there exists a natural numbers $\mathbb{N}$ such that

$$\sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ s,\ t > \mathbb{N}}}^{\infty} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}^s - x_{\sigma^m(k),\ \sigma^n(\ell)}^t\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon,$$

for $(k,\ \ell)$. Since for any fixed natural number M, we have from Definition 2.10,

$$\sum_{m \leq M} \sum_{\substack{n \leq M \\ s,\ t > \mathbb{N}}} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}^s - x_{\sigma^m(k),\ \sigma^n(\ell)}^t\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon,$$

for all $(k,\ \ell)$. By letting $t \to \infty$ in the above expression we obtain

$$\sum_{m \leq M} \sum_{\substack{n \leq M \\ s > \mathbb{N}}} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}^s - x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon.$$

Since M is arbitrary, by letting $M \to \infty$ we obtain

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k) + \sigma^n(\ell)\right)! \ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}^s - x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon,$$

for all $(k,\ \ell)$. Thus $g(x^s - x) \to 0$ as $s \to \infty$. Also (x^s) being a sequence in $\chi^2(A,\ f)$ be definition of $\chi^2(A,\ f)$ for each s with

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|x^s_{\sigma^m(k),\ \sigma^n(\ell)} - x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \to 0$$

as $(k,\ \ell) \to \infty$ thus $x \in \chi^2(A,\ f)$.

This completes the proof.

Theorem 3.3. Let $A = (a_{k\ell}^{mn})$ be nonnegative matrix such that

$$\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn}) < \infty$$

and let f be a modulus, then $\chi^2(A,\ f) \subset \Lambda^2(A,\ f)$.

Proof. Let $x \in \chi^2(A,\ f)$. Then by Definition 2.1 of (ii) and (iii) of the modulus function we granted the following:

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\\ \leq\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)} - 0\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) + f(|0|)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn}).\end{aligned}$$

There exists an integer N_p such that $|0| \leq N_p$. Thus we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn})\, f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\\ \leq\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)} - 0\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) + N_p f(1)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn}).\end{aligned}$$

Since

$$\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn}) < \infty$$

and $x \in \chi^2(A,\ f)$, we are granted $x \in \Lambda^2(A,\ f)$ and this completes the proof.

Theorem 3.4. Let $A = (a_{k\ell}^{mn})$ be nonnegative matrix such that

$$\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a_{k\ell}^{mn}) < \infty$$

and let f be a modulus, then $\Lambda^2(A) \subset \Lambda^2(A,\ f)$.

Proof. Let $x \in \Lambda^2(A)$, so that

$$\sup_{k,\ \ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn}\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Consider,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \\
= &\sum_{m=0}^{\infty} \sum_{\substack{n\,=\,0 \\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \leq \delta}}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \\
&+\sum_{m=0}^{\infty} \sum_{\substack{n\,=\,0 \\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} > \delta}}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right).
\end{aligned}$$

Then

$$\sum_{m=0}^{\infty} \sum_{\substack{n\,=\,0 \\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \leq \delta}}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \leq \epsilon \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn}. \tag{7}$$

For

$$\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} > \delta,$$

we use the fact that

$$\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \frac{\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta} < \left[1+\frac{\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta}\right],$$

where $[t]$ denoted the integer part of t and from conditions (ii) and (iii) of Definition 2.1, modulus function we have

$$\begin{aligned}
f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) &\leq \left[1+\frac{\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta}\right] f(1) \\
&\leq 2f(1)\,\frac{\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{\substack{n\,=\,0 \\ |xmn|^{\frac{1}{m+n}} > \delta}}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \\
\leq\ & \frac{2f(1)}{\delta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn}\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}.
\end{aligned}$$

which together with inequality (7) yields the following

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \\
\leq\ & \epsilon\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn} + \frac{2f(1)}{\delta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k\ell}^{mn}\left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}.
\end{aligned}$$

Since

$$\sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) < \infty$$

and $x \in \Lambda^2 (A)$, we are granted that $x \in \Lambda^2 (A,\ f)$ and this completes the proof.

Definition 3.1. Let f be modulus $a_{k\ell}^{mn}$ be a nonnegative RH-regular summability matrix method. Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 < p_{mn} < \sup\ p_{mn} = G$ and $D = \max \left(1,\ 2^{G-1}\right)$. Then for $a_{mn},\ b_{mn} \in \mathbb{N}$, the set of comples numbers for all $m,\ n \in \mathbb{N}$, we have

$$|a_{mn} + b_{mn}|^{\frac{1}{m+n}} \leq D \left\{ |a_{mn}|^{\frac{1}{m+n}} + |b_{mn}|^{\frac{1}{m+n}} \right\}.$$

Let $(X,\ q)$ be a semi normed space over the field $\mathbb{C}$ of complex numbers with the semi norm q. We define the following sequence spaces:

$$\chi^2 (A,\ f,\ p,\ q)$$
$$= x \in \chi^2 :\ P\text{-}\lim_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left[f \left(q \left((\sigma^m (k) + \sigma^n (\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{p_{mn}}$$
$$= 0,$$

$$\Lambda^2 (A,\ f,\ p,\ q)$$
$$= x \in \Lambda^2 :\ \sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left[f \left(q \left(\left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{p_{mn}}$$
$$< \infty.$$

Theorem 3.5. Let f_1 and f_2 be two modulus. Then $\chi^2 (A,\ f_1,\ p,\ q) \bigcap \chi^2 (A,\ f_2,\ p,\ q) \subseteq \chi^2 (A,\ f_1 + f_2,\ p,\ q)$.

Proof. The proof is easy so omitted.

Remark 3.1. Let f be a modulus q_1 and q_2 be two seminorm on X, we have

(i) $\chi^2 (A,\ f,\ p,\ q_1) \bigcap \chi^2 (A,\ f,\ p,\ q_2) \subseteq \chi^2 (A,\ f,\ p,\ q_1 + q_2)$,

(ii) If q_1 is stronger than q_2 then $\chi^2 (A,\ f,\ p,\ q_1) \subseteq \chi^2 (A,\ f,\ p,\ q_2)$,

(iii) If q_1 is equivalent to q_2 then $\chi^2 (A,\ f,\ p,\ q_1) = \chi^2 (A,\ f,\ p,\ q_2)$.

Theorem 3.6. Let $0 \leq p_{mn} \leq r_{mn}$ for all $m,\ n \in \mathbb{N}$ and let $\left\{ \frac{q_{mn}}{p_{mn}} \right\}$ be bounded. Then $\chi^2 (A,\ f,\ r,\ q) \subset \chi^2 (A,\ f,\ p,\ q)$.

Proof. Let

$$x \in \chi^2 (A,\ f,\ r,\ q),$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left[f \left(q \left((\sigma^m (k) + \sigma^n (\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{r_{mn}}. \tag{8}$$

Let

$$t_{mn} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left[f \left(q \left((\sigma^m (k) + \sigma^n (\ell))!\ \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{r_{mn}}, \tag{9}$$

we have $\gamma_{mn} = p_{mn}/r_{mn}$. Since $p_{mn} \le r_{mn}$, we have $0 \le \gamma_{mn} \le 1$. Let $0 < \gamma < \gamma_{mn}$. then

$$u_{mn} = \begin{cases} t_{mn}, & \text{if } (t_{mn} \ge 1), \\ 0, & \text{if } (t_{mn} < 1), \end{cases}$$

$$v_{mn} = \begin{cases} 0, & \text{if } (t_{mn} \ge 1), \\ t_{mn}, & \text{if } (t_{mn} < 1). \end{cases}$$

$$t_{mn} = u_{mn} + v_{mn}, \quad t_{mn}^{\gamma_{mn}} = u_{mn}^{\gamma_{mn}} + v_{mn}^{\gamma_{mn}}. \tag{10}$$

Now, it follows that

$$u_{mn}^{\gamma_{mn}} \le u_{mn} \le t_{mn}, \quad v_{mn}^{\gamma_{mn}} \le u_{mn}^{\gamma}. \tag{11}$$

Since

$$t_{mn}^{\gamma_{mn}} = u_{mn}^{\gamma_{mn}} + v_{mn}^{\gamma_{mn}}, \text{ we have } t_{mn}^{\gamma_{mn}} \le t_{mn} + v_{mn}^{\gamma}.$$

Thus,

$$\begin{aligned}
& \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right)^{r_{mn}} \right]^{\gamma_{mn}} \\
\le & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{r_{mn}}; \\
& \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right)^{r_{mn}} \right]^{p_{mn}/r_{mn}} \\
\le & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{r_{mn}}; \\
& \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{p_{mn}} \\
\le & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{r_{mn}}.
\end{aligned}$$

But

$$P\text{-}\lim_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{r_{mn}} = 0.$$

Therefore we have

$$P\text{-}\lim_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\ \sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{p_{mn}} = 0.$$

Hence

$$x \in \chi^2(A,\ f,\ p,\ q). \tag{12}$$

From (8) and (12) we get $x \in \chi^2(A,\ f,\ r,\ q) \subset x \in \chi^2(A,\ f,\ p,\ q)$.

Theorem 3.7. The space $x \in \chi^2(A,\ f,\ p,\ q)$ is solid and a such are monotone.

Proof. Let $(x_{mn}) \in x \in \chi^2(A,\ f,\ p,\ q)$ and (α_{mn}) be a sequence of scalars such that, $|\alpha_{mn}| \leq 1$ for all $m,\ n \in \mathbb{N}$. Then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\ \left|\alpha_{mn}x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{p_{mn}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{p_{mn}}\mathbb{N},$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\ \left|\alpha_{mn}x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{p_{mn}}$$
$$\leq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\ \left|x_{\sigma^m(k),\ \sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{p_{mn}}\mathbb{N},$$

for all $m,\ n \in \mathbb{N}$. This completes the proof.

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On the hybrid mean value of the Smarandache kn digital sequence with $SL(n)$ function and divisor function $d(n)$[1]

Le Huan

Department of Mathematics, Northwest University,
Xi'an, Shaanxi, P. R. China
E-mail: huanle1017@163.com

Abstract The main purpose of this paper is using the elementary method to study the hybrid mean value properties of the Smarandache kn digital sequence with $SL(n)$ function and divisor function $d(n)$, then give two interesting asymptotic formulae for it.

Keywords Smarandache kn digital sequence, $SL(n)$ function, divisor function, hybrid mean value, asymptotic formula.

§1. Introduction and results

For any positive integer k, the famous Smarandache kn-digital sequence $a(k,n)$ is defined as all positive integers which can be partitioned into two groups such that the second part is k times bigger than the first. For example, Smarandache $3n$ digital sequences $a(3,n)$ is defined as $\{a(3,n)\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots\}$, for example, $a(3,15) = 1545$. In the reference [1], Professor F. Smarandache asked us to study the properties of $a(k,n)$, about this problem, many people have studied and obtained many meaningful results. In [2], Lu Xiaoping studied the mean value of this sequence and gave the following theorem:

$$\sum_{n\leq N} \frac{n}{a(5,n)} = \frac{9}{50\ln 10} \cdot \ln N + O(1).$$

In [3], Gou Su studied the hybrid mean value of Smarandache kn sequence and divisor function $\sigma(n)$, and gave the following theorem:

$$\sum_{n\leq x} \frac{\sigma(n)}{a(k,n)} = \frac{3\pi^2}{k\cdot 20\cdot \ln 10} \cdot \ln x + O(1),$$

where $1 \leq k \leq 9$.

Inspired by the above conclusions, in this paper, we study the hybrid mean value properties of the Smarandache kn-digital sequence with $SL(n)$ function and divisor function $d(n)$, where

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$SL(n)$ is defined as the smallest positive integer k such that $n|[1,2,\ldots,k]$, that is $SL(n) = \min\{k : k \in N, n|[1,2,\ldots,k]\}$. And obtained the following results:

Theorem 1.1. Let $1 \le k \le 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} \frac{SL(n)}{a(k,n)} = \frac{3\pi^2}{k \cdot 20} \cdot \ln\ln x + O(1).$$

Theorem 1.2. Let $1 \le k \le 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} \frac{d(n) \cdot SL(n)}{a(k,n)} = \frac{\pi^4}{k \cdot 20} \cdot \ln\ln x + O(1).$$

§2. Lemmas

Lemma 2.1. For any real number $x > 1$, we have

$$\sum_{n \le x} \frac{SL(n)}{n} = \frac{\pi^2}{6} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Proof. For any real number $x > 1$, by reference [4] we have the asymptotic formula

$$\sum_{n \le x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Using Abel formula (see [6]) we get

$$\begin{aligned}
\sum_{1<n \le x} \frac{SL(n)}{n} &= \frac{1}{x}\left(\frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right)\right) + \int_1^x \frac{1}{t^2}\left(\frac{\pi^2}{12} \cdot \frac{t^2}{\ln t} + O\left(\frac{t^2}{\ln^2 t}\right)\right) dt \\
&= \frac{\pi^2}{12} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right) + \frac{\pi^2}{12}\int_1^x \frac{1}{\ln t}\, dt + O\left(\int_1^x \frac{1}{\ln^2 t}\, dt\right) \\
&= \frac{\pi^2}{6} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).
\end{aligned}$$

This proves Lemma 2.1.

Lemma 2.2. For any real number $x > 1$, we have

$$\sum_{n \le x} \frac{d(n) \cdot SL(n)}{n} = \frac{\pi^4}{18} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Proof. For any real number $x > 1$, by reference [5] we have the asymptotic formula

$$\sum_{n \le x} d(n) \cdot SL(n) = \frac{\pi^4}{36} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Using Abel formula (see [6]) we get

$$\begin{aligned}\sum_{1<n\le x}\frac{d(n)\cdot SL(n)}{n} &= \frac{1}{x}\left(\frac{\pi^4}{36}\cdot\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right)\right)+\int_1^x\frac{1}{t^2}\left(\frac{\pi^4}{36}\cdot\frac{t^2}{\ln t}+O\left(\frac{t^2}{\ln^2 t}\right)\right)\,dt\\ &= \frac{\pi^4}{36}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right)+\frac{\pi^4}{36}\int_1^x\frac{1}{\ln t}\,dt+O\left(\int_1^x\frac{1}{\ln^2 t}\,dt\right)\\ &= \frac{\pi^4}{18}\cdot\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right).\end{aligned}$$

This proves Lemma 2.2.

§3. Proof of the theorems

In this section, we shall use the elementary and combinational methods to complete the proof of our theorems. We just prove the case of $k=3$ and $k=5$, for other positive integers we can use the similar methods.

First we prove theorem 1.1. Let $k=3$, for any positive integer $x>3$, there exists a positive integer M such that

$$\underbrace{33\cdots33}_{M}<x\le\underbrace{33\cdots33}_{M+1}.$$

So

$$10^M-1<3x\le 10^{M+1}-1,$$

Then we have

$$\frac{\ln 3x}{\ln 10}-1-O\left(\frac{1}{10^M}\right)\le M<\frac{\ln 3x}{\ln 10}-O\left(\frac{1}{10^M}\right). \tag{1}$$

By the definition of $a(3,n)$ we have

$$\begin{aligned}\sum_{1\le n\le x}\frac{SL(n)}{a(3,n)} &= \sum_{n=1}^{3}\frac{SL(n)}{a(3,n)}+\sum_{n=4}^{33}\frac{SL(n)}{a(3,n)}+\sum_{n=34}^{333}\frac{SL(n)}{a(3,n)}+\cdots+\sum_{n=\frac{1}{3}\cdot10^{M-1}}^{\frac{1}{3}\cdot10^M-1}\frac{SL(n)}{a(3,n)}\\ &\quad+\sum_{\frac{1}{3}\cdot10^M\le n\le x}\frac{SL(n)}{a(3,n)}\\ &= \sum_{n=1}^{3}\frac{SL(n)}{n(10+3)}+\sum_{n=4}^{33}\frac{SL(n)}{n(10^2+3)}+\sum_{n=34}^{333}\frac{SL(n)}{n(10^3+3)}+\cdots\\ &\quad+\sum_{n=\frac{1}{3}\cdot10^{M-1}}^{\frac{1}{3}\cdot10^M-1}\frac{SL(n)}{n(10^{M+1}+3)}+\sum_{\frac{1}{3}\cdot10^M\le n\le x}\frac{SL(n)}{n(10^{M+2}+3)}.\end{aligned} \tag{2}$$

Form (1), (2) and lemma 2.1 we get

$$\begin{aligned}
\sum_{n=\frac{1}{3}\cdot 10^{k-1}}^{\frac{1}{3}\cdot 10^{k}-1} \frac{SL(n)}{n\cdot(10^k+3)} &= \sum_{\frac{1}{3}\cdot 10^{k}-1} \frac{SL(n)}{n\cdot(10^k+3)} - \sum_{n=\frac{1}{3}\cdot 10^{k-1}} \frac{SL(n)}{n\cdot(10^k+3)} \\
&= \frac{\pi^2}{6}\cdot\frac{\frac{1}{3}\cdot 10^k-\frac{1}{3}\cdot 10^{k-1}}{10^k+3}\cdot\frac{1}{\ln(\frac{1}{3}\cdot 10^k)}+O\left(\frac{1}{k^2}\right) \\
&= \frac{3\pi^2}{3\cdot 20}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right). \qquad (3)
\end{aligned}$$

Note that the identity $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\pi^2/6$ and the asymptotic formula

$$\sum_{1\le k\le M}\frac{1}{k}=\ln M+\gamma+O\left(\frac{1}{M}\right),$$

where γ is Euler's constant.
Form (1), (2) and (3) we get

$$\begin{aligned}
\sum_{1\le n\le x}\frac{SL(n)}{a(3,n)} &= \sum_{n=1}^{3}\frac{SL(n)}{a(3,n)}+\sum_{n=4}^{33}\frac{SL(n)}{a(3,n)}+\sum_{n=34}^{333}\frac{SL(n)}{a(4,n)}+\cdots+\sum_{n=\frac{1}{3}\cdot 10^{M-1}}^{\frac{1}{3}\cdot 10^{M}-1}\frac{SL(n)}{a(3,n)} \\
&\quad+\sum_{\frac{1}{3}\cdot 10^{M}\le n\le x}\frac{SL(n)}{a(3,n)} \\
&= \sum_{k=1}^{M}\frac{3\pi^2}{3\cdot 20}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right) \\
&= \frac{3\pi^2}{3\cdot 20}\ln\ln x+O(1).
\end{aligned}$$

Now we prove the case of $k=5$, for any positive integer $x>1$, there exists a positive integer M such that

$$\underbrace{200\cdots 00}_{M}<x\le\underbrace{199\cdots 99}_{M+1}.$$

So

$$10^M<5x\le 10^{M+1}-5,$$

Then we have

$$\frac{\ln 5x}{\ln 10}-1-O\left(\frac{1}{10^M}\right)\le M<\frac{\ln 5x}{\ln 10}. \qquad (4)$$

By the definition of $a(5,n)$ we have

$$\begin{aligned}
\sum_{1\le n\le x}\frac{SL(n)}{a(5,n)} &= \sum_{n=1}\frac{SL(n)}{a(5,n)}+\sum_{n=2}^{19}\frac{SL(n)}{a(5,n)}+\sum_{n=20}^{199}\frac{SL(n)}{a(5,n)}+\cdots+\sum_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1}\frac{SL(n)}{a(5,n)}\\
&\quad+\sum_{\frac{1}{5}\cdot 10^{M}\le n\le x}\frac{SL(n)}{a(5,n)}\\
&= \sum_{n=1}\frac{SL(n)}{n(10+5)}+\sum_{n=2}^{19}\frac{SL(n)}{n(10^2+5)}+\sum_{n=20}^{199}\frac{SL(n)}{n(10^3+5)}+\cdots\\
&\quad+\sum_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1}\frac{SL(n)}{n(10^{M+1}+5)}+\sum_{\frac{1}{5}\cdot 10^{M}\le n\le x}\frac{SL(n)}{n(10^{M+2}+5)}. \qquad (5)
\end{aligned}$$

Form (4), (5) and lemma 2.1 we get

$$\begin{aligned}
\sum_{n=\frac{1}{5}\cdot 10^{k-1}}^{\frac{1}{5}\cdot 10^{k}-1}\frac{SL(n)}{n\cdot(10^k+5)} &= \sum_{\frac{1}{5}\cdot 10^{k}-1}\frac{SL(n)}{n\cdot(10^k+5)}-\sum_{\frac{1}{5}\cdot 10^{k-1}}\frac{SL(n)}{n\cdot(10^k+5)}\\
&= \frac{\pi^2}{6}\cdot\frac{\frac{1}{5}\cdot 10^k-\frac{1}{5}\cdot 10^{k-1}}{10^k+5}\cdot\frac{1}{\ln(\frac{1}{5}\cdot 10^k)}+O\left(\frac{1}{k^2}\right)\\
&= \frac{3\pi^2}{5\cdot 20}\cdot\frac{1}{k}+O\left(\frac{1}{k^2}\right).
\end{aligned}$$

Similar to the proof $k=3$, we get

$$\begin{aligned}
\sum_{1\le n\le x}\frac{SL(n)}{a(5,n)} &= \sum_{n=1}\frac{SL(n)}{a(5,n)}+\sum_{n=2}^{19}\frac{SL(n)}{a(5,n)}+\sum_{n=20}^{199}\frac{SL(n)}{a(5,n)}+\cdots+\sum_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1}\frac{SL(n)}{a(5,n)}\\
&\quad+\sum_{\frac{1}{5}\cdot 10^{M}\le n\le x}\frac{SL(n)}{a(5,n)}\\
&= \sum_{k=1}^{M}\frac{3\pi^2}{5\cdot 20}\cdot\frac{1}{k}+O\left(\sum_{k=1}^{M}\frac{1}{k^2}\right)\\
&= \frac{3\pi^2}{5\cdot 20}\ln\ln x+O(1).
\end{aligned}$$

By using the same methods, we can also prove that the theorem holds for all integers $1\le k\le 9$. This completes the proof of theorem 1.1.

Similar to the proof of theorem 1.1, we can immediately prove theorem 1.2, we don't repeated here. As the promotion of this article, we can consider the hybrid mean value of Smarandache kn sequence with other functions such as $SL^*(n), Sdf(n), \sigma(S(n)), \Omega(S^*(n))$, and obtain the corresponding asymptotic formula.

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Iterations of strongly pseudocontractive maps in Banach spaces

Adesanmi Alao Mogbademu

Department of Mathematics, University of Lagos, Nigeria
E-mail: prinsmo@yahoo.com

Abstract Some convergence results for a family of strongly pseudocontractive maps is proved when at least one of the mappings is strongly pseudocontractive under some mild conditions, using a new iteration formular. Our results represent an improvement of some previously known results.

Keywords Three-step iteration, unified iteration, strongly pseudocontractive mapping, stro -ngly accretive operator, Banach spaces.
2000 AMS Mathematics Classification: 47H10, 46A03.

§1. Introduction

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of X and $\langle ., .\rangle$ denotes the generalized duality pairing.

Definition 1.1.[15] A mapping $T: X \to X$ with domain $D(T)$ and $R(T)$ in X is called strongly pseudocontractive if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le k\|x-y\|^2.$$

Closely related to the class of strongly pseudocontractive operators is the important class of strongly accretive operators. It is well known that T is strongly pseudo-contractive if and only if $(I-T)$ is strongly accretive, where I denotes the identity map. Therefore, an operator $T: X \to X$ is called strongly accretive if there exists a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \ge k\|x-y\|^2,$$

holds for all $x, y \in X$ and some $j(x-y) \in J(x-y)$. These operators have been studied and used by several authors (see, for example [13-20]).

The Mann iteration scheme [9], introduced in 1953, was used to prove the convergence of the sequence to the fixed points of mappings of which the Banach principle is not applicable. In 1974, Ishikawa [7] devised a new iteration scheme to establish the convergence of a Lipschitzian

pseudocontractive map when Mann iteration process failed to converge. Noor et al. [13], gave the following three-step iteration process for solving non-linear operator equations in real Banach spaces.

Let K be a nonempty closed convex subset of X and $T: K \to K$ be a mapping. For an arbitrary $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty} \subset K$, defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \ n \geq 0, \end{aligned} \tag{1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three sequences in $[0,1]$ for each n, is called the three-step iteration (or the Noor iteration). When $\gamma_n = 0$, then the three-step iteration reduces to the Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \ n \geq 0. \end{aligned} \tag{2}$$

If $\beta_n = \gamma_n = 0$, then (1) becomes the Mann iteration. It is the sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \geq 0. \tag{3}$$

Rafiq [15], recently studied the following of iterative scheme which he called the modified three-step iteration process, to approximate the unique common fixed points of a three strongly pseudocontractive mappings in Banach spaces.

Let $T_1, T_2, T_3: K \to K$ be three mappings. For any given $x_0 \in K$, the modified three-step iteration $\{x_n\}_{n=0}^{\infty} \subset K$ is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_2 z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T_3 x_n, \ n \geq 0, \end{aligned} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are three real sequences satisfying some conditions. It is clear that the iteration schemes (1)-(3) are special cases of (4).

It is worth mentioning that, several authors, for example, Xue and Fan [17], and Olaleru and Mogbademu [14] have recently used the iteration in equation (4) to approximate the common fixed points of some non-linear operators in Banach spaces.

In this paper, we study the following iterative scheme $f(T_1, v_n, x_n)$ involving a strong pseudocontraction T_1 and a sequence $\{v_n\}_{n=0}^{\infty}$ in X: $x_0 \in X$.

$$\begin{aligned} x_{n+1} &= f(T_1, v_n, x_n) \\ &= (1-\alpha_n)x_n + \alpha_n T_1 v_n, \quad n \geq 0, \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0, 1]$ to approximate the unique fixed point of a continuous strongly pseudocontractive map.

This iteration scheme is called unified iteration scheme because it unifies all the iterative schemes mentioned therein. For example:

(i) Its gives the Mann iteration as a special case when $v_n = x_n,\ (n \geq 0)$ and $T_1 = T$ (see Mann [9]).

(ii) If $v_n = y_n$ where $y_n = (1-\beta_n)x_n + \beta_n T_2 x_n$ and $T_1 = T_2 = T$ we have the Ishikawa iterative scheme (see, Ishikawa [7]).

(iii) If $v_n = y_n$ where $y_n = (1-\beta_n)x_n + \beta_n T_2 z_n$, $z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n$ and $T_1 = T_2 = T_3 = T$ we have the Noor iterative scheme defined in (1).

(iv) If $v_n = y_n$ where $y_n = (1-\beta_n)x_n + \beta_n T_2 z_n$, $z_n = (1-\gamma_n)x_n + \gamma_n T_3 x_n$ we have the modified Noor iterative scheme recently studied by Rafiq [15], Xue and Fan [17] and Olaleru and Mogbademu [14].

Moreover, we show that this unified iterative scheme can be used to approximate the common fixed points of a family of three maps. Thus, our results are natural generalization of several results.

In order to obtain the main results, the following Lemmas are needed.

Lemma 1.1.[15,16] Let E be real Banach space and $J:\ E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x,\ y \in E$

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y,\ j(x+y) >,\ \forall j(x+y) \in J(x+y).$$

Lemma 1.2.[16] Let (α_n) be a non-negative sequence which satisfies the following inequality

$$w_{n+1} \leq (1-\lambda_n)w_n + \delta_n,$$

where $\lambda_n \in (0,\ 1),\ \forall n \in N,\ \sum_{n=1}^{\infty} \lambda_n = \infty$ and $\delta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \alpha_n = 0$.

§2. Main results

Theorem 2.1. Let X be a real Banach space, K a non-empty, convex subset of X and let T_1 be a continuous and strongly pseudocontractive self mapping with pseudocontractive parameter $k \in (0,\ 1)$. For arbitrary $x_0 \in K$, let sequence $\{x_n\}_{n=0}^{\infty}$ be define by (5) where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,\ 1]$ satisfying the conditions:

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

If

$$\|T_1 v_n - T_1 x_{n+1}\| \to 0,$$

as $n \to \infty$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point of $T_1 \in K$.

Proof. The existence of a common fixed point follows from the result of Deimling (1978), and the uniqueness from the strongly pseudocontractivity of T_1. Since T_1 is strongly pseudocontractive, then there exists a constant k such that

$$\langle T_1 x - T_1 y,\ j(x-y)\rangle \leq k\|x-y\|^2.$$

Let ρ be such that $T_1\rho = \rho$. From Lemma 1.1, we have

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &= \langle x_{n+1}-\rho, j(x_{n+1}-\rho)\rangle \\
&= \langle (1-\alpha_n)x_n + \alpha_n T_1 v_n - (1-\alpha_n)\rho - \alpha_n, j(x_{n+1}-\rho)\rangle \\
&= \langle (1-\alpha_n)(x_n-\rho) + \alpha_n(T_1 v_n - \rho), j(x_{n+1}-\rho)\rangle \\
&= \langle (1-\alpha_n)(x_n-\rho), j(x_{n+1}-\rho)\rangle \\
&\quad + \langle \alpha_n(T_1 v_n - \rho), j(x_{n+1})\rangle \\
&= (1-\alpha_n)\langle x_n-\rho, j(x_{n+1}-\rho)\rangle \\
&\quad + \alpha_n\langle T_1 v_n - T_1 x_{n+1}, j(x_{n+1}-\rho)\rangle \\
&\quad + \alpha_n\langle T_1 x_{n+1} - \rho, j(x_{n+1}-\rho)\rangle. \qquad (6)
\end{aligned}$$

By strongly pseudocontractivity of T_1, we get

$$\alpha_n\langle T_1 x_{n+1} - \rho,\ j(x_{n+1}) \leq \alpha_n k\|x_{n+1}-\rho\|^2,$$

for each $j(x_{n+1}-\rho) \in J(x_{n+1}-\rho)$, and a constant $k \in (0,\ 1)$.

From inequality (6) and inequality $ab \leq \frac{a^2+b^2}{2}$, we obtain that

$$(1-\alpha_n)\|x_n-\rho\|\|x_{n+1}-\rho\| \leq \frac{1}{2}(1-\alpha_n)\|x_n-\rho\|^2 + \|x_{n+1}-\rho\|^2 \qquad (7)$$

and

$$\alpha_n\|T_1 v_n - T_1 x_{n+1}\|\|x_{n+1}-\rho\| \leq \frac{1}{2}(\|T_1 v_n - T_1 x_{n+1}\|^2 + \alpha_n^2\|x_{n+1}-\rho\|^2). \qquad (8)$$

Substituting (7) and (8) into (6), we infer that

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq \frac{1}{2}((1-\alpha_n)^2\|x_n-\alpha_n\|^2 + \|x_{n+1}-\rho\|^2) \\
&\quad + \frac{1}{2}(\|T_1 v_n - T_1 x_{n+1}\|^2 + \alpha_n^2\|x_{n+1}-\rho\|^2) \\
&\quad + \alpha_n k\|x_{n+1}-\rho\|^2. \qquad (9)
\end{aligned}$$

Multiplying inequality (9) by 2 throughout, we have

$$\begin{aligned}
2\|x_{n+1}-\rho\|^2 &\leq (1-\alpha_n)^2\|x_n-\rho\|^2 + \|x_{n+1}-\rho\|^2 + \|T_1^n y_n - T_1^n x_{n+1}\|^2 \\
&\quad + \alpha^2\|x_{n+1}-\rho\|^2 + 2\alpha_n k\|x_{n+1}-\rho\|^2.
\end{aligned}$$

By collecting like terms $\|x_{n+1}-\rho\|^2$ and simplifying, we have

$$(1-2\alpha_n k - \alpha_n^2)\|x_{n+1}-\rho\|^2 \leq (1-\alpha_n)^2\|x_n-\rho\|^2 + \|T_1 v_n - T_1 x_{n+1}\|^2.$$

Since $\lim_{n\to\infty}[1-2\alpha_n k - \alpha_n^2] = 1 > 0$, there exists a positive integer N_0 such that

$1-2\alpha_n k-\alpha_n^2>0$ for $n\geq N_0$. Then the inequality above implies that

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq \frac{(1-\alpha_n)^2}{1-2\alpha_n k-\alpha_n^2}\|x_n-\rho\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1+\frac{(1-2\alpha_n+\alpha_n^2-1+2k\alpha_n+\alpha_n^2)}{1-2\alpha_n k-\alpha_n^2})\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&= (1-\frac{2\alpha_n((1-k)+\alpha_n)}{1-2\alpha_n k-\alpha_n^2})\|x_n-\rho\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-\frac{2\alpha_n(1-k)}{1-2\alpha_n k-\alpha_n^2})\|x_n-\rho\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-2\alpha_n(1-k))\|x_n-\rho\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-\alpha_n(1-k))\|x_n-\rho\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}. \qquad (10)
\end{aligned}$$

It follows from (10) that,

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n r)\|x_n-p\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2},\quad \forall n\geq n_0, \qquad (11)$$

where $r=(1-k)\in(0,\ 1)$. Put $\lambda_n=r\alpha_n$, $w_n=\|x_n-\rho\|^2$, $\delta_n=\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}$.

Thus, Lemma 1.2 ensures that $\lim_{n\to\infty}\|x_n-p\|=0$. This completes the proof.

Theorem 2.2. Let X, T_1 and $\{x_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Suppose $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ satisfying conditions of Theorem 2.1 with

$$\alpha_n\geq m>0,\ \forall n\geq 0,$$

where m is a constant. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T_1 and

$$\begin{aligned}
\|x_n-\rho\|^2 &\leq (1-mr)\|x_{n-1}-\rho\|^2+M\\
&\leq (1-mr)\|x_0-\rho\|^2+\tfrac{(1-(1-mr)^n)}{mr}M,
\end{aligned}$$

for all $n\geq 0$, which implies that

$$\|x_n-\rho\|\leq((1-mr)\|x_0-\rho\|^2+\frac{(1-(1-mr)^n)}{mr}M)^{\frac{1}{2}},$$

where $M=sup\frac{1}{1-2\alpha_n k-\alpha_n^2}\|T_1v_n-T_1x_{n+1}\|^2$.

Proof. As in the proof of Theorem 2.1, we conclude that $F(T_1)=p$ and

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n r)\|x_n-p\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-mr)\|x_n-p\|^2+\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-mr)\|x_n-p\|^2+M. \qquad (12)
\end{aligned}$$

Put $\lambda_n=mr$, $w_n=\|x_n-\rho\|^2$, $\delta_n=M=\frac{\|T_1v_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}$. Thus, Lemma 1.2 ensures that $\lim_{n\to\infty}\|x_n-p\|=0$.

Observe from inequality (12) that,

$$\begin{aligned}
\|x_1-p\|^2 &\leq (1-mr)\|x_0-p\|^2+M,\\
\|x_2-p\|^2 &\leq (1-mr)\|x_1-p\|^2+M\\
&\leq (1-mr)[(1-mr)\|x_0-p\|^2+M]+M\\
&= (1-mr)^2\|x_0-p\|^2+M+M(1-mr),\\
&\quad .\\
&\quad .\\
&\quad .\\
\|x_n-p\|^2 &\leq (1-mr)^n\|x_0-\rho\|^2+\tfrac{(1-(1-mr)^n)}{mr}M,
\end{aligned}$$

for all $n\geq 0$, which implies that

$$\|x_n-\rho\|\leq((1-mr)^n\|x_0-\rho\|^2+\frac{(1-(1-mr)^n)}{mr}M)^{\frac{1}{2}}.$$

This completes the proof.

Theorem 2.3. Let X be a real Banach space, K a non-empty, convex subset of X and let $T_1,\ T_2,\ T_3$ be continuous. Suppose T_1 is strongly pseudocontractive self mapping with pseudocontractive parameter $k\in(0,\ 1)$. For arbitrary $x_0\in K$, let sequence $\{x_n\}_{n=0}^{\infty}$ be define by (4) where $\{\alpha_n\}_{n=0}^{\infty},\ \{\beta_n\}_{n=0}^{\infty},\ \{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,\ 1]$ satisfying the conditions:

$$\lim_{n\to\infty}\alpha_n=0,\ \sum_{n=1}^{\infty}\alpha_n=\infty.$$

If

$$\|T_1y_n-T_1x_{n+1}\|\to 0,$$

as $n\to\infty$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique common fixed point of $T_1,\ T_2,\ T_3$.

Proof. If we set $v_n=y_n$ in Theorem 2.1. In this case , the desired result follows immediately from Theorem 2.1.

Remark 2.1. Theorem 2.3 improves and extends Theorem 2.1 of Xue and Fan (2008) among others in the following ways:

(i) It abolishes the condition that $T_i(K)$ is bounded.

(ii) At least one member of the family of $T_1,\ T_2,\ T_3$ is strongly pseudocontractive.

(iii) We obtained a convergence rate estimate in Theorem 2.2.

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Generalized convolutions for special classes of harmonic univalent functions

D. O. Makinde

Department of Mathematics,
Obafemi Awolowo University, Ile-Ife, Nigeria
E-mail: domakinde.comp@gmail.com

Abstract In this paper, we further investigate the preservation of close-to-convex of certain harmonic functions under convolution.
Keywords Convolution, harmonic, convex, close-to-convex.

§1. Introduction and preliminaries

Let H denote the family of continuous complex-valued functions which are harmonic in the open unit disk $U = \{z \in C : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. Clunie and Sheil-Small in [2] developed the basic theory of harmonic functions $f \in H$ which are univalent in U and have the normalization $f(0) = 0,\ f_z(0) = 1$. Such function may be written as $f = h + \overline{g}$, where h and g are members of A. f is sense-preserving if $|g'| < |h'|$ in U, or equivalently if the dilation function $w = \frac{h'}{g'}$ satisfies $|w(z)| < 1$ for $z \in U$. We may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

Let S_H denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U, where h and g are in A and of the form (1).

For harmonic function $f = h + \overline{g}$, we call the h the analytic part and $\overline{g}$ the co-analytic part of f. The class S_H reduce to the class S normalized analytic univalent functions in U if the co-analytic part of f is zero. We denote by $K_H,\ S_H,\ C_H$, the subclass of S_H consisting of harmonic functions which are respectively convex, starlike, and close-to-convex in U. A function is said to be starlike, convex and close-to-convex in U if it maps each $|z| = r < 1$ onto a starlike, convex and close-to-convex domain respectively.

Let the convolution of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n + \sum_{n=1}^{\infty} \overline{b}_{1n} \overline{z}^n \text{ and } f_2(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n + \sum_{n=1}^{\infty} \overline{b}_{2n} \overline{z}^n$$

be defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1n} a_{2n} z^n + \sum_{n=1}^{\infty} \bar{b}_{1n} \bar{b}_{2n} \bar{z}^n.$$

It is noted that the above convolution formula is reduced to the Hadamard product if the co-analytic part of the functions $f_1(z)$ and $f_2(z)$ is identically zero. On the convolution of complex-valued harmonic functions, Ruschweyh and Shell-Small [3] proved the following:

Lemma 1.1. Let ϕ and ψ be convex analytic in U. Then, we have the following:

(i) $(\phi * \psi)(z)$ is convex analytic in U,

(ii) $(\phi * f)(z)$ is close-to-convex analytic in U if f is close-to-convex analytic in U,

(iii) $(\phi * zf')/(\phi * z\psi')$ takes all its values in a convex domain D if f'/ψ' takes all its values in D.

Lemma 1.2. [2] Let $h,\ g \in A$,

(i) If $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close-to-convex analytic in U for each $\epsilon(|\epsilon| = 1)$, then $f = h + \overline{g} \in C_H$,

(ii) If $h + \epsilon g$ is harmonic and locally univalent in U and if $h + \epsilon g$ is convex analytic in U for some $\epsilon(|\epsilon| \leq 1)$, then $f = h + \overline{g} \in C_H$.

Ahuja and Jaharigiri [1] proved that:

Lemma 1.3. Let g and h be analytic in U such that $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close-to-convex in U for each $\epsilon(|\epsilon| = 1)$ and suppose ϕ is convex analytic in U, then

$$(\varphi + \overline{\sigma}\phi) * (h + \overline{g}) \in C_H, |\sigma| = 1.$$

Lemma 1.4. [1] Suppose h and ϕ are convex analytic in U and g is analytic in U with $|g'(z)| < |h'(z)|$ for each $z \in U$. Then $(\phi + \epsilon\overline{\phi}) * (h + \overline{g}) \in C_H$ for each $|\epsilon| \leq 1$.

§2. Main results

Let $h^k(z) = z^k + \sum_{n=k+1} a_n z^{kn},\ g^k(z) = \sum_{n=1} b_n z^{kn},\ k \in N$, we now present the main results.

Theorem 2.1. Let h and g be analytic in U such that $|(g^k)'(0)| < |(h^k)'(0)$ and $h^k + \varepsilon g^k$ is close-to-convex in U for each $\varepsilon(|\varepsilon| = 1$. If ϕ is convex analytic in U and $\phi^k \leq \phi;\ h^k \leq h$, then $(\phi^k + (\overline{\gamma\phi}^k)) * (h^k + \overline{g}^k) \subset C_H,\ |\gamma| = 1,\ k \in N$.

Proof.

$$\begin{aligned} (\phi^k + (\overline{\gamma\phi}^k)) * (h^k + \overline{g}^k) &= (\phi^k * h^k) + ((\overline{\gamma\phi})^k * \overline{g}^k) \\ &= H^k + \overline{G}^k. \end{aligned}$$

But

$$\begin{aligned} |(G^k)'(0)| &= |((\gamma\phi)^k * g^k)'|_{z=0} = |\frac{1}{z}\phi^k * \gamma(g^k)'|_{z=0} < |\frac{1}{z}\phi^k * \gamma(h^k)'|_{z=0} \\ &= |(\phi^k * \gamma h^k)'| \leq |(\phi * \gamma h)'|_{z=0}. \end{aligned}$$

Using lemma 1.1 (ii) , we obtain for each $\alpha = \frac{\varepsilon}{\gamma}$ that

$$H^k + \alpha G^k = (\phi^k * h^k) + (\alpha\gamma\phi^k * g^k) = \phi^k * (h^k + \varepsilon g^k).$$

Thus by the hypothesis that $h^k + \epsilon g^k$ is close to convex and using lemma 1.1 (ii) we have $H^k + \alpha G^k$ is close-to-convex analytic in U.

This concludes the proof of theorem 2.1.

Theorem 2.2. Let h and ϕ be convex analytic in U and g is analytic in U such that $|g'(z)| < |h'(z)|$. Let $g^k \leq g,\ h^k \leq h$. Then for each $|\varepsilon| \leq 1,\ (\phi^k + (\varepsilon\overline{\phi})^k) * (h^k + \overline{g^k})$ is close to convex. $k \in N$.

Proof. We write $(\phi^k + (\varepsilon\overline{\phi})^k) * (h^k + \overline{g^k}) = (\phi^k * h^k) + ((\overline{\overline{\varepsilon}\phi})^k * \overline{g^k}) = H^k + \overline{G^k}$. By lemma 1.1 (i) ϕ and h being convex analytic in U implies $H = \phi * h$ is convex in U.

But

$$\left|\frac{(G^k)'}{(H^k)'}\right| \leq \left|\frac{(\phi^k * (g^k))'}{(\phi^k * h^k)'}\right| = \left|\frac{\frac{1}{z}\phi^k * (g^k)'}{\frac{1}{z}\phi^k * (h^k)'}\right|.$$

By lemma 1.4, ϕ being convex and $\left|\frac{g'}{h'}\right| < 1$ implies that $\left|\frac{\phi * zg'}{\phi * zh'}\right| < 1$. But $\left|\frac{\phi^k * z(g^k)'}{\phi^k * z(h^k)'}\right| \leq \left|\frac{\phi * zg'}{\phi * zh'}\right| < 1$.

This shows that $H^k + \overline{G}^k$ is locally univalent in U. Thus, $H^k + \overline{G}^k = (\phi^k + (\varepsilon\overline{\phi})^k) * (h^k + (\overline{g})')$ is close-to-convex in U.

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On soft fuzzy C structure compactification

V. Visalakshi[†], M. K. Uma[‡] and E. Roja[♯]

Department of Mathematics, Sri Sarada College for Women,
Salem, 636016, Tamil Nadu, India
E-mail: visalkumar_cbe@yahoo.co.in

Abstract In this paper the concept of soft fuzzy $T^{'}$-prefilter, soft fuzzy $T^{'}$-ultrafilter, soft fuzzy prime $T^{'}$-prefilter are introduced. The concept of soft fuzzy F^{*} space and soft fuzzy normal family are discussed. The concept of soft fuzzy C space is established. Also the process of structure compactification of soft fuzzy C space is established.

Keywords Soft fuzzy $T^{'}$-prefilter, soft fuzzy $T^{'}$-ultrafilter, soft fuzzy prime $T^{'}$-prefilter, soft fuzzy F^{*} space, soft fuzzy normal family, soft fuzzy C structure, soft fuzzy C space, soft fuzz -y C structure prefilter.
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§1. Introduction

Zadeh [6] introduced the fundamental concepts of fuzzy sets in his classical paper. Fuzzy sets have applications in many fields such as information [3] and control [4]. In mathematics, topology provided the most natural framework for the concepts of fuzzy sets to flourish. Chang [2] introduced and developed the concept of fuzzy topological spaces. The concept of soft fuzzy topological space is introduced by Ismail U. Tiryaki [5]. N. Blasco Mardones, M. Macho stadler and M. A. de Prada [1] introduced the concept of fuzzy closed filter, fuzzy δ^{c}-ultrafilter and the compactification of a fuzzy topological space.

In this paper soft fuzzy $T^{'}$-prefilter, soft fuzzy $T^{'}$-ultrafilter and soft fuzzy prime $T^{'}$-prefilter are introduced and studied. Some of their properties are discussed. Soft fuzzy F^{*} space and soft fuzzy $T^{'}$-normal family are established and their properties are discussed. The soft fuzzy C structure in soft fuzzy C space is introduced. Also the process of soft fuzzy structure compactification using soft fuzzy $T^{'}$-prefilter has been established.

§2. Preliminaries

Definition 2.1.[5] Let X be a set, μ be a fuzzy subset of X and $M \subseteq X$. Then, the pair (μ, M) will be called a soft fuzzy subset of X. The set of all soft fuzzy subsets of X will be denoted by $SF(X)$.

Definition 2.2.[5] The relation $\sqsubseteq$ on $SF(X)$ is given by $(\mu, M) \sqsubseteq (\gamma, N) \Leftrightarrow (\mu(x) < \gamma(x))$ or $(\mu(x) = \gamma(x)$ and $x \notin M/N)$, $\forall\, x \in X$ and for all $(\mu, M),\ (\gamma, N) \in SF(X)$.

Property 2.1.[5] If $(\mu_j, M_j)_{j\in J} \in SF(X)$, then the family $\{(\mu_j, M_j)|\, j \in J\}$ has a meet, that is greatest lower bound, in $(SF(X), \sqsubseteq)$, denoted by

$$\sqcap_{j\in J}(\mu_j, M_j) \text{ such that } \sqcap_{j\in J}(\mu_j, M_j) = (\mu, M),$$

where

$$\mu(x) = \bigwedge_{j\in J} \mu_j(x),\ \forall\, x \in X;\quad M = \bigcap_{j\in J} M_j.$$

Property 2.2.[5] If $(\mu_j, M_j)_{j\in J} \in SF(X)$, then the family $\{(\mu_j, M_j)|\, j \in J\}$ has a join, that is least upper bound, in $(SF(X), \sqsubseteq)$, denoted by

$$\sqcup_{j\in J}(\mu_j, M_j) \text{ such that } \sqcup_{j\in J}(\mu_j, M_j) = (\mu, M),$$

where

$$\mu(x) = \bigvee_{j\in J} \mu_j(x), \forall\, x \in X;\quad M = \bigcup_{j\in J} M_j.$$

Definition 2.3.[5] For $(\mu, M) \in SF(X)$ the soft fuzzy set $(\mu, M)^{'} = (1-\mu, X|M)$ is called the complement of (μ, M).

Definition 2.4.[5] Let $x \in X$ and $S \in I$ define x_s: $X \to I$ by

$$x_s(z) = \begin{cases} s, & \text{if } z = x, \\ 0, & \text{otherwise.} \end{cases}$$

Then the soft fuzzy set $(x_s, \{x\})$ is called the point of $SF(X)$ with base x and value s.

Definition 2.5.[5] The soft fuzzy point $(x_r, \{x\}) \sqsubseteq (\mu, M)$ is denoted by $(x_r, \{x\}) \in (\mu, M)$ and refer to $(x_r, \{x\})$ is an element of (μ, M).

Property 2.3.[5] Let $\varphi : X \to Y$ be a point function.

(i) The mapping $\varphi^{\rightharpoonup}$ from $SF(X)$ to $SF(Y)$ corresponding to the image operator of the difunction (f, F) is given by

$$\varphi^{\rightharpoonup}(\mu, M) = (\gamma, N), \text{ where } \gamma(y) = \sup\{\mu(x)/y = \varphi(x)\} \text{ and } N = \{\varphi(x)/x \in M\}.$$

(ii) The mapping $\varphi^{\leftharpoonup}$from $SF(Y)$ to $SF(X)$ corresponding to the inverse image of the difunction (f, F) is given by

$$\varphi^{\leftharpoonup}(\gamma, N) = (\gamma \circ \varphi, \varphi^{-1}[N]).$$

Note. $\varphi^{\rightharpoonup}(\mu, M) = \varphi(\mu, M)$ and $\varphi^{\leftharpoonup}(\gamma, N) = \varphi^{-1}(\gamma, N)$.

Definition 2.6.[5] A subset $T \subseteq SF(X)$ is called an SF-topology on X if

(i) $(0, \phi)$ and $(1, X) \in T$.

(ii) $(\mu_j, M_j) \in T,\ j = 1,\ 2,\ 3, \cdots, n \Rightarrow \sqcap_{j=1}^{n}(\mu_j, M_j) \in T$.

(iii) $(\mu_j, M_j),\ j \in J \Rightarrow \sqcup_{j\in J}(\mu_j, M_j) \in T$.

The elements of T are called soft fuzzy open, and those of $T^{'} = \{(\mu, M)/(\mu, M)^{'} \in T\}$ is called soft fuzzy closed.

If T is a SF-topology on X we call the pair (X, T) a SF-topological space (in short, $SFTS$).

Definition 2.7.[5] Let T be an SF-topology on X and V an SF-topology on Y. Then a function $\varphi : X \to Y$ is called $T - V$ continuous if $(\nu, N) \in V \Rightarrow \varphi^{\leftarrow}(\nu, N) \in T$.

Note. $T - V$ continuous is denoted by soft fuzzy continuous.

Definition 2.8.[5] A soft fuzzy topological space (X, T) is said to be a soft fuzzy compact if whenever $\sqcup_{i\in I}(\lambda_i, M_i) = (1, X)$, $(\lambda_i, M_i) \in T$, $i \in I$, there is a finite subset J of I with $\sqcup_{j\in J}(\lambda_j, M_j) = (1, X)$.

§3. Soft fuzzy closed filter theory

Definition 3.1. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F} \subset T^{'}$ satisfying the following conditions:

(i) $\mathcal{F}$ is nonempty and $(0, \phi) \notin \mathcal{F}$.

(ii) If (μ_1, M_1), $(\mu_2, M_2) \in \mathcal{F}$, then $(\mu_1, M_1) \sqcap (\mu_2, M_2) \in \mathcal{F}$.

(iii) If $(\mu, M) \in \mathcal{F}$ and $(\lambda, N) \in T^{'}$ with $(\mu, M) \sqsubseteq (\lambda, N)$, then $(\lambda, N) \in \mathcal{F}$.

$\mathcal{F}$ is called a soft fuzzy closed filter or a soft fuzzy $T^{'}$-prefilter on X.

Definition 3.2. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F}$ be a soft fuzzy $T^{'}$-prefilter and let $\mathcal{B} \subset \mathcal{F}$. $\mathcal{B}$ is called a soft fuzzy base for $\mathcal{F}$ if for each $(\mu, M) \in \mathcal{F}$ there is $(\nu, P) \in \mathcal{B}$ such that $(\nu, P) \sqsubseteq (\mu, M)$.

Definition 3.3. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{H} \subset T^{'}$. $\mathcal{H}$ is a soft fuzzy subbase for some soft fuzzy $T^{'}$-prefilter if the collection $\{(\mu_1, M_1) \sqcap (\mu_2, M_2) \sqcap \cdots \sqcap (\mu_n, M_n)/(\mu_i, M_i) \in \mathcal{H}\}$ is a soft fuzzy base for some soft fuzzy $T^{'}$-prefilter.

Property 3.1. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{B} \subset T^{'}$. Then the following statements are equivalent:

(i) There is a unique soft fuzzy $T^{'}$-prefilter $\mathcal{F}$ such that $\mathcal{B}$ is a soft fuzzy base for it.

(ii) (a) $\mathcal{B}$ is nonempty and $(0, \phi) \notin \mathcal{B}$.

(b) If (ν_1, P_1), $(\nu_2, P_2) \in \mathcal{B}$, there is $(\nu_3, P_3) \in \mathcal{B}$ with $(\nu_3, P_3) \sqsubseteq (\nu_1, P_1) \sqcap (\nu_2, P_2)$.

Proof. Proof follows from Definition 3.1, Definition 3.2 and Definition 3.3.

Definition 3.4. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{B}$ be a soft fuzzy base satisfying the conditions (a) and (b) in Property 3.1. Then the generated soft fuzzy $T^{'}$-prefilter $\mathcal{F}$ is defined by

$$\mathcal{F} = \{(\mu, M) \in T^{'}/\exists(\nu, P) \in \mathcal{B} \text{ with } (\nu, P) \sqsubseteq (\mu, M)\}.$$

Definition 3.5. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{G} \subset T^{'}$ with the intersection of any finite subcollection from $\mathcal{G}$ is nonempty. There exists a unique soft fuzzy $T^{'}$-prefilter containing $\mathcal{G}$, whose base is the set of all the finite intersections of elements in $\mathcal{G}$. Such a soft fuzzy $T^{'}$-prefilter is called the soft fuzzy $T^{'}$-prefilter generated by $\mathcal{G}$.

Property 3.2. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F}$ be a soft fuzzy $T^{'}$-prefilter and $(\mu, M) \in T^{'}$. The following statements are equivalent:

(i) $\mathcal{F} \cup \{(\mu, M)\}$ is contained in a soft fuzzy $T^{'}$-prefilter.

(ii) For each $(\nu, P) \in \mathcal{F}$, we have $(\mu, M) \sqcap (\nu, P) \neq (0, \phi)$.

Proof. Proof is simple.

Definition 3.6. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F}$ be a soft fuzzy $T^{'}$-prefilter. $\mathcal{F}$ is a soft fuzzy $T^{'}$-ultrafilter if $\mathcal{F}$ is a maximal element in the set of soft fuzzy $T^{'}$-prefilters ordered by the inclusion relation.

Property 3.3. Let (X, T) be a soft fuzzy topological space. Every soft fuzzy $T^{'}$-prefilter is contained in some soft fuzzy $T^{'}$-ultrafilter.

Proof. Proof is obvious from the above definition.

Property 3.4. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F}$ be a soft fuzzy $T^{'}$-prefilter on X. Then the following statements are equivalent:

(i) $\mathcal{F}$ is a soft fuzzy $T^{'}$-ultrafilter.

(ii) If (μ, M) is an element of $T^{'}$ such that $(\mu, M) \sqcap (\nu, P) \neq (0, \phi)$ for each $(\nu, P) \in \mathcal{F}$, then $(\mu, M) \in \mathcal{F}$.

(iii) If $(\mu, M) \in T^{'}$ and $(\mu, M) \notin \mathcal{F}$, then there is $(\nu, P) \in \mathcal{F}$ such that $(\mu, M) \sqcap (\nu, P) = (0, \phi)$.

Proof. (i)$\Rightarrow$(ii) Suppose $(\mu, M) \in T^{'}$ and $(\mu, M) \sqcap (\nu, P) \neq (0, \phi)$, for each $(\nu, P) \in \mathcal{F}$. By Property 3.2, there is a soft fuzzy $T^{'}$-prefilter $\mathcal{F}^*$ generated by $\mathcal{F} \cup \{(\mu, M)\}$. Then $\mathcal{F} \subset \mathcal{F}^*$. Since $\mathcal{F}$ is a soft fuzzy $T^{'}$-ultrafilter, $\mathcal{F}$ is the maximal element. Hence $\mathcal{F} = \mathcal{F}^*$, that is $(\mu, M) \in \mathcal{F}$.

(ii)$\Rightarrow$(iii) Let $(\mu, M) \in T^{'}$ and suppose $(\mu, M) \notin \mathcal{F}$. By assumption, there exists $(\nu, P) \in \mathcal{F}$ such that $(\mu, M) \sqcap (\nu, P) = (0, \phi)$.

(iii)$\Rightarrow$(i) Let $\mathcal{G}$ be a soft fuzzy $T^{'}$-prefilter with $\mathcal{F} \subset \mathcal{G}$ and $\mathcal{F} \neq \mathcal{G}$. Let $(\mu, M) \in \mathcal{G}$ such that $(\mu, M) \notin \mathcal{F}$. By assumption, there is $(\nu, P) \in \mathcal{F}$ such that $(\nu, P) \sqcap (\mu, M) = (0, \phi)$. Since $(\nu, P),\ (\mu, M) \in \mathcal{G}$, $(\nu, P) \sqcap (\mu, M) \in \mathcal{G}$ implies that $(0, \phi) \in \mathcal{G}$, which is a contradiction. Hence $\mathcal{F} = \mathcal{G}$. Thus $\mathcal{F}$ is a maximal soft fuzzy $T^{'}$-prefilter. Hence $\mathcal{F}$ is a soft fuzzy $T^{'}$-ultrafilter.

Property 3.5. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be a pair of different soft fuzzy $T^{'}$-ultrafilters on X. Then $(\sqcap_i(\mu_i, M_i)) \sqcap (\sqcap_j(\mu_j, M_j)) = (0, \phi)$ for all $(\mu_i, M_i) \in \mathcal{U}_1$ and $(\mu_j, M_j) \in \mathcal{U}_2$.

Proof. Suppose $(\sqcap_i(\mu_i, M_i)) \sqcap (\sqcap_j(\mu_j, M_j)) \neq (0, \phi)$. Then for some $x,\ y \in X$ $(\wedge_i \mu_i(x) \wedge \wedge_j \mu_j(x)) > 0$ and $y \in \cap_i M_i \cap \cap_j M_j$.

$\Rightarrow \mu_i(x) \wedge \mu_j(x) > 0$ and $y \in M_i \cap M_j$ for all i and j.

$\Rightarrow (\mu_i, M_i) \sqcap (\mu_j, M_j) \neq (0, \phi)$.

By the above property $(\mu_i, M_i) \in \mathcal{U}_2$ and $(\mu_j, M_j) \in \mathcal{U}_1$ for all i and j. Thus $\mathcal{U}_1 = \mathcal{U}_2$. Which is a contradiction to our hypothesis. Hence $(\sqcap_i(\mu_i, M_i)) \sqcap (\sqcap_j(\mu_j, M_j)) = (0, \phi)$ for all $(\mu_i, M_i) \in \mathcal{U}_1$ and $(\mu_j, M_j) \in \mathcal{U}_2$.

Definition 3.7. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F}$ be a soft fuzzy $T^{'}$-prefilter on X. $\mathcal{F}$ is said to be a soft fuzzy prime $T^{'}$-prefilter, if given $(\mu, M),\ (\nu, P) \in T^{'}$ such that $(\mu, M) \sqcup (\nu, P) \in \mathcal{F}$, then $(\mu, M) \in \mathcal{F}$ or $(\nu, P) \in \mathcal{F}$.

Property 3.6. Let (X, T) be a soft fuzzy topological space. Every soft fuzzy $T^{'}$-ultrafilter $\mathcal{U}$ on X is a soft fuzzy prime $T^{'}$-prefiter.

Proof. Let $(\mu, M),\ (\nu, P) \in T^{'}$ such that $(\mu, M) \sqcup (\nu, P) \in \mathcal{U}$. Suppose $(\mu, M),\ (\nu, P) \notin \mathcal{U}$. Then there exist $(\mu_1, M_1),\ (\nu_1, P_1) \in \mathcal{U}$ with $(\mu, M) \sqcap (\mu_1, M_1) = (0, \phi)$ and $(\nu, P) \sqcap (\nu_1, P_1) = (0, \phi)$. Since $(\mu, M) \sqcup (\nu, P),\ (\mu_1, M_1),\ (\nu_1, P_1) \in \mathcal{U}$. Since $\mathcal{U}$ is a soft fuzzy $T^{'}$-ultrafilter,

$(\mu, M) \sqcup (\nu, P) \sqcap (\mu_1, M_1) \sqcap (\nu_1, P_1) \in \mathcal{U}$.

$$\begin{aligned} & ((\mu, M) \sqcup (\nu, P)) \sqcap (\mu_1, M_1) \sqcap (\nu_1, P_1) \\ = & ((\mu, M) \sqcap (\mu_1, M_1)) \sqcup ((\nu, P) \sqcap (\mu_1, M_1)) \sqcap (\nu_1, P_1) \\ = & ((0, \phi) \sqcap (\nu_1, P_1)) \sqcup ((0, \phi) \sqcap (\mu_1, M_1)) \\ = & (0, \phi). \end{aligned}$$

Which is a contradiction to our assumption $\mathcal{U}$ is a soft fuzzy T'-ultrafilter. Hence (μ, M) or $(\nu, P) \in \mathcal{U}$. Thus $\mathcal{U}$ is a soft fuzzy prime T'-prefilter.

Property 3.7. Let (X, T) be a soft fuzzy topological space. Let $\mathcal{F}$ be a soft fuzzy T'-prefilter. Let $\mathcal{P}(\mathcal{F})$ be the family of all soft fuzzy prime T'-prefilters which contains $\mathcal{F}$. Then $\mathcal{F} = \cap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}$.

Proof. Let $\mathcal{F}$ be a soft fuzzy T'-prefilter. Let $\mathcal{P}(\mathcal{F})$ be the family of all soft fuzzy prime T'-prefilters which contains $\mathcal{F}$. Therefore

$$\mathcal{F} \subset \cap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G}. \tag{1}$$

Let $(\mu, \ M) \in T'$ such that $(\mu, M) \notin \mathcal{F}$. Consider the family $\mathcal{L} = \{\mathcal{G} \ / \ \mathcal{G}$ is a soft fuzzy T'-prefilter, $\mathcal{F} \subset \mathcal{G}$ and $(\mu, M) \notin \mathcal{G}\}$. $\mathcal{L}$ is an inductive set, therefore there exist maximal elements. Let $\mathcal{U}$ be the maximal element in $\mathcal{L}$.

Let $(\lambda_1, N_1), \ (\lambda_2, N_2) \in T'$ with $(\lambda_1, N_1) \sqcup (\lambda_2, N_2) \in \mathcal{U}$ such that $(\lambda_1, N_1), \ (\lambda_2, N_2) \notin \mathcal{U}$. Let the family $\mathcal{J} = \{(\nu, P) \in T' / (\nu, P) \sqcup (\lambda_2, N_2) \in \mathcal{U}\}$.

(i) Since $(\lambda_1, N_1) \in T'$ and $(\lambda_1, N_1) \sqcup (\lambda_2, N_2) \in \mathcal{U}$, which implies that $(\lambda_1, N_1) \in \mathcal{J}$. Hence $\mathcal{J}$ is nonempty.

Suppose $(0, \phi) \in \mathcal{J}$, By definition of $\mathcal{J}$, $(\lambda_2, N_2) \in \mathcal{U}$. Which is a contradiction. Hence $(0, \phi) \notin \mathcal{J}$.

(ii) If $(\nu_1, P_1), \ (\nu_2, P_2) \in \mathcal{J}$. By definition of $\mathcal{J}$, $(\nu_1, P_1) \sqcup (\lambda_2, N_2) \in \mathcal{U}$ and $(\nu_2, P_2) \sqcup (\lambda_2, N_2) \in \mathcal{U}$. Since $\mathcal{U}$ is a soft fuzzy T'-prefilter, $[(\nu_1, P_1) \sqcup (\lambda_2, N_2)] \sqcap [(\nu_2, N_2) \sqcup (\lambda_2, N_2)] \in \mathcal{U}$.

$\Rightarrow [(\nu_1, P_1) \sqcap (\nu_2, N_2)] \sqcup (\lambda_2, N_2) \in \mathcal{U}$.

$\Rightarrow [(\nu_1, P_1) \sqcap (\nu_2, N_2)] \in \mathcal{J}$.

(iii) If $(\nu, P) \in \mathcal{J}$ and $(\lambda, N) \in T'$ such that $(\nu, P) \sqsubseteq (\lambda, N)$. Since $(\nu, P) \sqsubseteq (\lambda, N)$, $(\nu, P) \sqcup (\lambda_2, N_2) \sqsubseteq (\lambda, N) \sqcup (\lambda_2, N_2)$. Since $(\nu, P) \sqcup (\lambda_2, N_2) \in \mathcal{U}$ and $\mathcal{U}$ is a soft fuzzy T'-prefilter, $(\lambda, N) \sqcup (\lambda_2, N_2) \in \mathcal{U}$. Which implies $(\lambda, N) \in \mathcal{J}$. Thus $\mathcal{J}$ is a soft fuzzy T'-prefilter.

If $(\nu, P) \in \mathcal{U}$, $(\nu, P) \sqcup (\lambda_2, N_2) \in \mathcal{U}$. Which implies $(\nu, P) \in \mathcal{J}$. Thus $\mathcal{U} \subset \mathcal{J}$. Since $(\lambda_1, N_1) \in \mathcal{J}$ and $(\lambda_1, N_1) \notin \mathcal{U}$. Thus $\mathcal{U} \neq \mathcal{J}$.

Now let $\mathcal{K} = \{(\lambda, N) \in T' / (\mu, M) \sqcup (\lambda, N) \in \mathcal{U}\}$.

(i) Suppose $(0, \phi) \in \mathcal{K}$, then by definition of $\mathcal{K}$, $(\mu, M) \in \mathcal{U}$. Which is a contradiction to our assumption $\mathcal{U} \in \mathcal{L}$ and $(\mu, M) \notin \mathcal{U}$. Hence $(0, \phi) \notin \mathcal{K}$. Since $(1, X) \in \mathcal{U}$, which implies $(1, X) \in \mathcal{K}$. Thus $\mathcal{K}$ is nonempty and $(0, \phi) \notin \mathcal{K}$.

(ii) If $(\lambda^*, N^*), \ (\lambda^{**}, N^{**}) \in \mathcal{K}$. By definition of $\mathcal{K}$, $(\lambda^*, N^*) \sqcup (\mu, M) \in \mathcal{U}$ and $(\lambda^{**}, N^{**}) \sqcup (\mu, M) \in \mathcal{U}$. Since $\mathcal{U}$ is a soft fuzzy T'-prefilter,$((\lambda^*, N^*) \sqcap (\lambda^{**}, N^{**}) \sqcup (\mu, M) \in \mathcal{U}$. Therefore $(\lambda^*, N^*) \sqcap (\lambda^{**}, N^{**}) \in \mathcal{K}$.

(iii) If $(\lambda, N) \in \mathcal{K}$ and $(\lambda^*, N^*) \in T'$ such that $(\lambda^*, N^*) \sqsupseteq (\lambda, N)$, then $(\lambda^*, N^*) \in \mathcal{K}$. Hence $\mathcal{K}$ is a soft fuzzy T'-prefilter.

Now,

(i) $\mathcal{F} \subset \mathcal{K}$ follows from $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{U} \subset \mathcal{K}$.

(ii) $(\mu, M) \notin \mathcal{K}$ for $(\mu, M) \notin \mathcal{U}$. Thus $\mathcal{K} \in \mathcal{L}$ and $\mathcal{U} \subset \mathcal{K}$. Maximality of $\mathcal{U}$ implies that $\mathcal{U} = \mathcal{K}$. Suppose $(\mu, M) \in \mathcal{J}$. Then $(\mu, M) \sqcup (\lambda_2, N_2) \in \mathcal{U}$ which implies $(\lambda_2, N_2) \in \mathcal{K} = \mathcal{U}$. Which implies $(\lambda_2, N_2) \in \mathcal{U}$. Which is a contradiction to our assumption $(\lambda_2, N_2) \notin \mathcal{U}$. Thus $(\mu, M) \notin \mathcal{J}$. Since $\mathcal{F} \subset \mathcal{J}$ and $(\mu, M) \notin \mathcal{J}$, we have $\mathcal{J} \in \mathcal{L}$ and since $\mathcal{U} \subset \mathcal{J}$. $\mathcal{U} \neq \mathcal{J}$. Which is a contradiction to the maximality of $\mathcal{U}$. Thus $(\lambda_1, N_1),\ (\lambda_2, N_2) \in \mathcal{U}$. Therefore $\mathcal{U}$ is a soft fuzzy prime $T^{'}$-prefilter and $(\mu, M) \notin \mathcal{U}$. Thus

$$\cap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G} \subset \mathcal{F}. \tag{2}$$

From (1) and (2), it follows that $\cap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G} = \mathcal{F}$.

Property 3.8. The following are equivalent for a soft fuzzy topological space (X, T).

(i) (X, T) is soft fuzzy compact.

(ii) For any family of soft fuzzy closed sets $\{(\lambda_i, M_i)\}_{i \in J}$ with the property that $\sqcap_{j \in F}(\lambda_j, M_j) \neq (0, \phi)$ for any finite subset F of J, we have $\sqcap_{i \in J}(\lambda_i, M_i) \neq (0, \phi)$.

Proof. (i) $\Rightarrow$(ii) Assume that (X, T) is soft fuzzy compact. Let $\{(\lambda_i, M_i)\}_{i \in J}$ be the family of soft fuzzy closed sets with the property $\sqcap_{j \in F}(\lambda_j, M_j) \neq (0, \phi)$ for any finite subset F of J. Let $(\mu_i, N_i) = (1, X) - (\lambda_i, M_i)$. The collection $\{(\mu_i, N_i)\}_{i \in J}$ is a family of all soft fuzzy open sets in (X, T). If $(\lambda_1, M_1),\ (\lambda_2, M_2),\ (\lambda_3, M_3), \cdots, (\lambda_n, M_n)$ are finite number of soft fuzzy closed sets in $\{(\lambda_i, M_i)\}_{i \in J}$, then

$$\sqcap_{i=1}^{n}(\lambda_i, M_i) = (1, X) - \sqcup_{i=1}^{n}(\mu_i, N_i).$$

Similarly

$$\sqcap_{i \in J}(\lambda_i, M_i) = (1, X) - \sqcup_{i \in J}(\mu_i, N_i).$$

By hypothesis $\sqcap_{i=1}^{n}(\lambda_i, M_i) \neq (0, \phi)$. Therefore $\sqcup_{i=1}^{n}(\mu_i, N_i) \neq (1, X)$. Since (X, T) is soft fuzzy compact, $\sqcup_{i \in J}(\mu_i, N_i) \neq (1, X)$. Thus $\sqcap_{i \in J}(\lambda_i, M_i) \neq (0, \phi)$.

(ii) $\Rightarrow$(i) Assume that (X, T) is not soft fuzzy compact. Let $\{(\mu_i, N_i)\}_{i \in J}$ be a family of soft fuzzy open sets which is a cover for (X, T). Since (X, T) is not soft fuzzy compact, there is no finite subset F of J such that $\{(\mu_j, N_j)\}_{j \in F}$ is a cover for (X, T), that is $\sqcup_{j \in F}(\mu_j, N_j) \neq (1, X)$.

Now,

$$\begin{aligned} \sqcap_{j \in F}((1, X) - (\mu_j, N_j)) &= (1, X) - \sqcup_{j \in F}(\mu_j, N_j) \\ &\neq (0, \phi). \end{aligned}$$

By hypothesis, $\sqcap_{i \in J}((1, X) - (\mu_i, N_i)) \neq (0, \phi)$. This implies $\sqcup_{i \in J}(\mu_i, N_i) \neq (1, X)$, which is a contradiction. Hence (X, T) is soft fuzzy compact.

Property 3.9. For a soft fuzzy topological space (X, T), the following are equivalent:

(i) (X, T) is soft fuzzy compact.

(ii) Every soft fuzzy $T^{'}$-prefilter $\mathcal{F}$ satiafies $\sqcap_{(\mu, M) \in \mathcal{F}}(\mu, M) \neq (0, \phi)$.

(iii) Every soft fuzzy prime $T^{'}$-prefilter $\mathcal{F}$ satiafies $\sqcap_{(\mu, M) \in \mathcal{F}}(\mu, M) \neq (0, \phi)$.

(iv) Every soft fuzzy $T^{'}$-ultrafilter $\mathcal{U}$ satiafies $\sqcap_{(\mu, M) \in \mathcal{U}}(\mu, M) \neq (0, \phi)$.

Proof. (i)$\Rightarrow$ (ii) Suppose $\sqcap_{(\mu,M)\in\mathcal{F}}(\mu, M) = (0, \phi)$. Then $\sqcup_{(\mu,M)\in\mathcal{F}}[(1, X) - (\mu, M)] = (1, X)$. Since $(1, X) - (\mu, M) \in T$ and (X, T) is soft fuzzy compact, there exist finite subcollection, $\{(1, X) - (\mu_1, M_1),\ (1, X) - (\mu_2, M_2),\ \cdots,\ (1, X) - (\mu_n, M_n)\}$ such that $(1, X) - (\mu_1, M_1) \sqcup (1, X) - (\mu_2, M_2) \sqcup \cdots \sqcup (1, X) - (\mu_n, M_n) = (1, X)$. Which implies $(\mu_1, M_1) \sqcap (\mu_2, M_2) \sqcap \cdots \sqcap (\mu_n, M_n) = (0, \phi)$. Which is a contradiction to our assumption $\mathcal{F}$ is a soft fuzzy T'-prefilter. Hence $\sqcap_{(\mu,M)\in\mathcal{F}}(\mu, M) \neq (0, \phi)$.

(ii)$\Rightarrow$(iii) Suppose that every soft fuzzy prime T'-prefilter $\mathcal{F}$ satisfies $\sqcap_{(\mu,M)\in\mathcal{F}}(\mu, M) = (0, \phi)$. Since every soft fuzzy prime T'-prefilter is soft fuzzy T'-prefilter. Which is a contradiction. Hence every soft fuzzy prime T'-prefilter $\mathcal{F}$ satisfies $\sqcap_{(\mu,M)\in\mathcal{F}}(\mu, M) \neq (0, \phi)$.

(iii)$\Rightarrow$(iv) Suppose that every soft fuzzy T'-ultrafilter $\mathcal{U}$ satiafies $\sqcap_{(\mu,M)\in\mathcal{U}}(\mu, M) = (0, \phi)$. Since every soft fuzzy T'-ultrafilter is soft fuzzy prime T'-prefilter. Which is a contradiction. Hence every soft fuzzy T'-ultrafilter $\mathcal{U}$ satisfies $\sqcap_{(\mu,M)\in\mathcal{F}}(\mu, M) \neq (0, \phi)$.

(iv)$\Rightarrow$(i) Suppose $\mathcal{H}$ is a family of soft fuzzy closed sets on X with finite intersection property. For each $(\nu, P) \in \mathcal{H}$ the family $\mathcal{G}_{(\nu,P)} = \{(\mu, M) \in T'/(\mu, M) \sqsupseteq (\nu, P)\}$. Then $(\nu, P) \in \mathcal{G}_{(\nu,P)}$. Let $\mathcal{G} = \cup_{(\nu,P)\in\mathcal{H}}\mathcal{G}_{(\nu,P)}$. Since $\mathcal{H}$ has a finite intersection property, $\mathcal{G}$ has finite intersection property. Thus $\mathcal{H}$ and $\mathcal{G}$ are soft fuzzy T'-prefilters. Thus there exists a soft fuzzy T'-ultrafilter $\mathcal{U}$ such that $\mathcal{H} \subset \mathcal{G} \subset \mathcal{U}$. Hence $\sqcap_{(\mu,M)\in\mathcal{U}}(\mu, M) \sqsubseteq \sqcap_{(\mu,M)\in\mathcal{G}}(\mu, M) \sqsubseteq \sqcap_{(\mu,M)\in\mathcal{H}}(\mu, M)$. Since $\sqcap_{(\mu,M)\in\mathcal{U}}(\mu, M) \neq (0, \phi)$. Thus $\sqcap_{(\mu,M)\in\mathcal{H}}(\mu, M) \neq (0, \phi)$. Hence $\mathcal{H}$ is a family of soft fuzzy closed sets on X with finite intersection property and $\sqcap_{(\mu,M)\in\mathcal{H}}(\mu, M) \neq (0, \phi)$. Which implies (X, T) is soft fuzzy compact space.

Definition 3.8. Let (X, T) be a soft fuzzy topological space. Let $(x_s, \{x\})$ be any soft fuzzy point. The nonempty collection $\mathcal{F}_{(x_s,\{x\})} = \{(\mu, M) \in T'/(x_s, \{x\}) \in (\mu, M)\}$ is a soft fuzzy prime T'-prefilter on X, then $\mathcal{F}_{(x_s,\{x\})}$ is called soft fuzzy T'-prefilter generated by $(x_s, \{x\})$.

Definition 3.9. Let (X, T) be any soft fuzzy topological space. The collection T' is said to be a soft fuzzy normal family if given $(\mu_1, M_1),\ (\mu_2, M_2) \in T'$ such that $(\mu_1, M_1) \sqcap (\mu_2, M_2) = (0, \phi)$, there exist $(\nu_1, N_1),\ (\nu_2, N_2) \in T'$ with $(\nu_1, N_1) \sqcup (\nu_2, N_2) = (1, X)$, $(\mu_1, M_1) \sqcap (\nu_1, N_1) = (0, \phi)$ and $(\mu_2, M_2) \sqcap (\nu_2, N_2) = (0, \phi)$.

Property 3.10. Let (X, T) be any soft fuzzy topological space and T' be a soft fuzzy normal family. Every soft fuzzy prime T'-prefilter $\mathcal{F}$ is contained in a unique soft fuzzy T'-ultrafilter.

Proof. Let $\mathcal{F}$ be a soft fuzzy prime T'-prefilter. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be soft fuzzy T'-ultrafilters such that $\mathcal{F} \subset \mathcal{U}_1$ and $\mathcal{F} \subset \mathcal{U}_2$. Suppose $\mathcal{U}_1 \neq \mathcal{U}_2$. Then there exist $(\mu_1, M_1) \in \mathcal{U}_1$ and $(\mu_2, M_2) \in \mathcal{U}_2$ with $(\mu_1, M_1) \sqcap (\mu_2, M_2) = (0, \phi)$. Since T' is a normal family, there exist $(\nu_1, N_1),\ (\nu_2, N_2) \in T'$ with $(\nu_1, N_1) \sqcup (\nu_2, N_2) = (1, X)$, $(\mu_1, M_1) \sqcap (\nu_1, N_1) = (0, \phi)$ and $(\mu_2, M_2) \sqcap (\nu_2, N_2) = (0, \phi)$. Since $(\nu_1, N_1) \sqcup (\nu_2, N_2) = (1, X)$ and $\mathcal{F}$ is a soft fuzzy prime T'-prefilter, $(\nu_1, N_1) \in \mathcal{F}$ or $(\nu_2, N_2) \in \mathcal{F}$. Suppose if $(\nu_1, N_1) \in \mathcal{F}$, then $(\nu_1, N_1) \in \mathcal{U}_1$ and $(\nu_1, N_1) \in \mathcal{U}_2$. Thus $(\mu_1, M_1) \sqcap (\nu_1, N_1) = (0, \phi)$ with $(\mu_1, M_1),\ (\nu_1, N_1) \in \mathcal{U}_1$. Which is a contradiction. Similarly $(\nu_2, N_2) \in \mathcal{F}$, then $(\nu_2, N_2) \in \mathcal{U}_1$ and $(\nu_2, N_2) \in \mathcal{U}_2$. Thus $(\mu_2, M_2) \sqcap (\nu_2, N_2) = (0, \phi)$ with $(\mu_2, M_2),\ (\nu_2, N_2) \in \mathcal{U}_2$. Which is a contradiction. Hence $\mathcal{U}_1 = \mathcal{U}_2$. Thus every soft fuzzy prime T'-prefilter $\mathcal{F}$ is contained in a unique soft fuzzy T'-ultrafilter.

Remark 3.1. Let (X,T) be any soft fuzzy topological space and $T^{'}$ be a soft fuzzy normal family. For every $x \in X$, the soft fuzzy point $(x_s,\{x\})$, there exists a unique soft $T^{'}$-ultrafilter $\mathcal{U}_{(x_s,\{x\})}$ which contains $\mathcal{F}_{(x_s,\{x\})}$.

Proof. Let $(x_s,\{x\})$ be any soft fuzzy point. Then $\mathcal{F}_{(x_s,\{x\})}$ is a soft fuzzy $T^{'}$-prefilter generated by $(x_s,\{x\})$. By Definition 3.8 and Property 3.10, $\mathcal{F}_{(x_s,\{x\})}$ is a soft fuzzy prime $T^{'}$-prefilter contained in a unique soft fuzzy $T^{'}$-ultrafilter.

Remark 3.2. Let (X,T) be any soft fuzzy topological space. If $(x_s,\{x\})$ and $(y_t,\{y\})$ be any two soft fuzzy points with $x=y$ then $\mathcal{U}_{(x_s,\{x\})} = \mathcal{U}_{(y_t,\{y\})}$.

Proof. Let (X,T) be a soft fuzzy topological space. Let $(x_s,\{x\})$ and $(y_t,\{y\})$ be two soft fuzzy points with $x=y$. Assume that $\mathcal{U}_{(x_s,\{x\})} \neq \mathcal{U}_{(y_t,\{y\})}$. By Property 3.5 $(\sqcap_i(\mu_i,M_i)) \sqcap (\sqcap_j(\mu_j,M_j)) = (0,\phi)$ for all $(\mu_i,M_i) \in \mathcal{U}_{(x_s,\{x\})}$ and $(\mu_j,M_j) \in \mathcal{U}_{(y_t,\{y\})}$. Thus $(x_s,\{x\}) \in \mathcal{U}_{(x_s,\{x\})}$ and $(y_t,\{y\}) \in \mathcal{U}_{(y_t,\{y\})}$. Since $x=y$, $(x_s,\{x\}) \sqcap (y_t,\{y\}) \neq (0,\phi)$. Which is a contradiction. Hence $\mathcal{U}_{(x_s,\{x\})} = \mathcal{U}_{(y_t,\{y\})}$.

Definition 3.10. Let (X,T) be any soft fuzzy topological space. For each $x \in X$, the collection of soft fuzzy points $\mathcal{P}_x = \{(x_s,\{x\})/s \in (0,1]\}$. For each $(x_s,\{x\}) \in \mathcal{P}_x$ the only soft fuzzy $T^{'}$-ultrafilter which contains $\mathcal{F}_{(x_s,\{x\})}$ is denoted by $\mathcal{U}^x$.

Definition 3.11. A soft fuzzy topological space (X,T) is said to be a soft fuzzy F^* space, if for each $x \in X$ there exists a minimum value $s \in (0,1]$ such that the soft fuzzy point $(x_s,\{x\})$ belongs to $T^{'}$.

Proprety 3.11. Let (X,T) be any soft fuzzy F^* space and $T^{'}$ be a soft fuzzy normal family. For each $x \in X$, the soft fuzzy $T^{'}$-ultrafilter $\mathcal{U}^x$, $\sqcap_{(\mu_i,M_i)\in\mathcal{U}^x}(\mu_i,M_i)$ is atmost a soft fuzzy point $(x_s,\{x\})$.

Proof. Let $x \in X$. Since (X,T) is a soft fuzzy F^* space, there exists a minimum value $s \in (0,1]$ such that $(x_s,\{x\}) \in T^{'}$. By Definition 3.10, $(x_s,\{x\}) \in \mathcal{U}^x$. Hence $\sqcap_{(\mu_i,M_i)\in\mathcal{U}^x}(\mu_i,M_i) = (x_s,\{x\})$.

§4. C structure compactification of a soft fuzzy C space

Let (X,T) be a non-compact soft fuzzy topological space. Let $\gamma(X)$ be the collection of all soft fuzzy $T^{'}$-ultrafilters on X. Suppose (X,T) is a soft fuzzy F^* space and $T^{'}$ is a soft fuzzy normal family. Associated with each $(\mu,M) \in T^{'}$, we define $(\mu,M)^* = (\mu^*,M^*)$ where $\mu^* \in I^{\gamma(X)}$ and $M^* \subset \gamma(X)$. For each $\mathcal{U} \in \gamma(X)$,

$$(\mu^*(\mathcal{U}),M^*) = \begin{cases} (0,\phi), & \text{if } \mathcal{U} \neq \mathcal{U}^x,\ \forall x \in X \text{ and } (\mu,M) \notin \mathcal{U},\ M=\phi, \\ (1,\gamma(X)), & \text{if } \mathcal{U} \neq \mathcal{U}^x,\ \forall x \in X \text{ and } (\mu,M) \in \mathcal{U},\ M=X, \\ (\mu(x),\mathcal{U}^x), & \text{if } \exists\ x \in X \text{ with } \mathcal{U}=\mathcal{U}^x,\ x \in M. \end{cases}$$

Property 4.1. Under the previous conditions, the following identities hold:
(i) $(0^*,\phi^*)_X = (0,\phi)_{\gamma(X)}$.
(ii) $(1^*,X^*)_X = (1,\gamma(X))$.
(iii) $(x_s,\{x\})^*_X = (\mathcal{U}^x_s,\{\mathcal{U}^x\})$.
Proof. Proof is obvious from the above definition of (μ^*,M^*).
Notation 4.1. For each $\mathcal{U}^x \in \gamma(X)$, the soft fuzzy point is denoted by $(\mathcal{U}^x_s,\{\mathcal{U}^x\})$.

Definition 4.1. A soft fuzzy C structure (in short, $SFCst$) consists of all soft fuzzy sets of the form (μ^*, M^*). A soft fuzzy C space $(\gamma(X), SFCst)$ is a space which admits soft fuzzy $\mathcal{C}$ structure.

Definition 4.2. A soft fuzzy C closure of a soft fuzzy set in soft fuzzy C space is defined by

$cl_{Cst}(\mu^*, M^*) = \sqcap\{(\nu^*, P^*)/(\nu^*, P^*) \sqsupseteq (\mu^*, M^*) \text{ and } (\nu^*, P^*) \in SF\mathcal{C}st\}$.

Property 4.2. Let $e: X \to \gamma(X)$ defined by $e(x) = \mathcal{U}^x \forall x \in X$, $e(1, X)$ is soft fuzzy C dense in soft fuzzy C space $(\gamma(X), SFCst)$ that is $cl_{Cst}(e(1, X)) = (1, \gamma(X))$.

Proof. Let (μ, M) be a soft fuzzy set with $\mu \in I^X$ and $M \subseteq X$. Now $(e(\mu), e(M))$ is a soft fuzzy set in soft fuzzy C space with $e(\mu) \in I^{\gamma(X)}$ and $e(M) \subset \gamma(X)$ and it is defined for each $\mathcal{U} \in \gamma(X)$ as follows:

$$(e(\mu)(\mathcal{U}), e(M)) = \begin{cases} (\mu(x), \mathcal{U}^x), & \text{if } \exists\, x \in X \ni \mathcal{U} = \mathcal{U}^x \text{ and } x \in M, \\ (0, \phi), & \text{if } \mathcal{U} \neq \mathcal{U}^x,\ \forall x \in X \text{ and } M = \phi. \end{cases}$$

Let $C = cl_{Cst}(e(1, X))$. We know that $e(1, X) \sqsubseteq C$, then for each $x \in X$ and $\mathcal{U}^x \in \gamma(X)$, $(1, \gamma(X)) \sqsubseteq C$. Therefore $C = (1, \gamma(X))$. Hence $cl_{Cst}(e(1, X)) = (1, \gamma(X))$. Thus $e(1, X)$ is soft fuzzy C dense in soft fuzzy C space.

Definition 4.3. Let (X, T) be a soft fuzzy topological space and $(\gamma(X), SFCst)$ be a soft fuzzy C space. Then $f: (X, T) \to (\gamma(X), SFCst)$ is a soft fuzzy C *continuous** function, if the inverse image of every soft fuzzy set in $(\gamma(X), SFCst)$ is soft fuzzy closed in (X, T).

Definition 4.4. Let (X, T) be a soft fuzzy topological space and $(\gamma(X), SFCst)$ is a soft fuzzy C space. Then $f: (X, T) \to (\gamma(X), SFCst)$ is said to be Soft fuzzy C *closed** function, if the image of every soft fuzzy closed set in (X,T) is a soft fuzzy set in $(\gamma(X), SFCst)$.

Definition 4.5. Let (X, T) be a soft fuzzy topological space and $(\gamma(X), SFCst)$ is a soft fuzzy C space. If $f: (X, T) \to (\gamma(X), SFCst)$ is an injective soft fuzzy C *continuous** and f is soft fuzzy C *closed**. Then f is a soft fuzzy C *embedding**.

Property 4.3. The function e is a soft fuzzy C *embedding** of X into $\gamma(X)$.

Proof. (i) Let $x,\ y \in X$ and $x \neq y$ then $\mathcal{U}^x \neq \mathcal{U}^y$. Let $(x_s, \{x\})$ and $(y_q, \{y\})$ be two distinct soft fuzzy points.

(a) If $(x_s, \{x\}) \neq (y_q, \{y\})$ for each $\mathcal{U} \in \gamma(X)$,

$$(e(x_s)(\mathcal{U}), e(x)) = \begin{cases} (\mathcal{U}_q^x, \{\mathcal{U}^x\}), & \text{if } \exists\, x \in X \ni \mathcal{U} = \mathcal{U}^x, \\ (0, \phi), & \text{if } \mathcal{U} \neq \mathcal{U}^x,\ \forall x \in X \text{ and } \{x\} = \phi. \end{cases}$$

Similarly,

$$(e(y_q)(\mathcal{U}), e(\{y\})) = \begin{cases} (\mathcal{U}_q^y, \{\mathcal{U}^y\}), & \text{if } \exists\, y \in X \ni \mathcal{U} = \mathcal{U}^y, \\ 0, & \text{if } \mathcal{U} \neq \mathcal{U}^y,\ \forall y \in X \text{ and } \{y\} = \phi. \end{cases}$$

Therefore $e(x_s, \{x\}) \neq e(y_q, \{y\})$.

(b) If $x = y$ and $s \neq q$. For each $\mathcal{U}^x \in \gamma(X)$,

$$\begin{aligned} e(x_s, \{x\}) &= (\mathcal{U}_s^x, \{\mathcal{U}^x\}) \\ &= (\mathcal{U}_s^y, \{\mathcal{U}^y\}) \\ &\neq (\mathcal{U}_q^y, \{\mathcal{U}^y\}) \\ &\neq e(y_q, \{y\}). \end{aligned}$$

Hence e is soft fuzzy C one to one function.

(ii) For each $(\mu^*, M^*) \in SFCst$ and $x \in X$,

$$\begin{aligned} e^{-1}(\mu^*(x), M^*) &= (e^{-1}(\mu^*(x)), e^{-1}(M^*)) \\ &= (\mu^* \circ e(x), e^{-1}(M^*)) \\ &= (\mu(\mathcal{U}^x), e^{-1}(M^*)) \\ &= (\mu(x), M). \end{aligned}$$

Thus $e^{-1}(\mu^*, M^*) = (\mu, M)$. Thus the inverse image of every soft fuzzy set in $SFCst$ is soft fuzzy closed in (X, T). Hence e is soft fuzzy$C$ *continuous** function.

(iii) For each $\mathcal{U} \in \gamma(X)$ and $(\mu, M) \in T'$,

$$e(\mu(\mathcal{U}), M) = (e(\mu)(\mathcal{U}), e(M)) = \begin{cases} (\mu(x), \mathcal{U}^x), & \text{if } \exists\ x \in X \ni \mathcal{U} = \mathcal{U}^x \text{ and } x \in M, \\ (0, \phi), & \text{if } \mathcal{U} \neq \mathcal{U}^x,\ \forall x \in X \text{ and } M = \phi. \end{cases}$$

Therefore $e(\mu, M)$ is a soft fuzzy set in $(\gamma(X), SFCst)$. Hence e is soft fuzzy C *closed** function.

Definition 4.6. Let $(\gamma(X), SFCst)$ be a soft fuzzy C space. Let$\mathcal{F} \subset SFCst$ satisfying the following conditions:

(i) $\mathcal{F}$ is nonempty and $(0, \phi) \notin \mathcal{F}$.

(ii) If $(\mu_1^*, M_1^*),\ (\mu_2^*, M_2^*) \in \mathcal{F}$, then $(\mu_1^*, M_1^*) \sqcap (\mu_2^*, M_2^*) \in \mathcal{F}$.

(iii) If $(\mu^*, M^*) \in \mathcal{F}$ and $(\lambda^*, N^*) \in SFCst$ with $(\mu^*, M^*) \sqsubseteq (\lambda^*, N^*)$, then $(\lambda^*, N^*) \in \mathcal{F}$.

$\mathcal{F}$ is called a soft fuzzy $SFCst$-prefilter (in short, $(T_{\gamma(X)})'$- prefilter) on $\gamma(X)$.

Definition 4.7. A soft fuzzy C space $(\gamma(X), SFCst)$ is said to be a soft fuzzy C *compact** space if whenever $\sqcup_{i\in I}(\lambda_i^*, M_i^*) = (1, \gamma(X))$, $(\lambda_i^*, M_i^*) \in SFCst$, $i \in I$, there is a finite subset J of I with $\sqcup_{j\in J}(\lambda_j^*, M_j^*) = (1, \gamma(X))$.

Definition 4.8. A soft fuzzy C space $(\gamma(X), SFCst)$ is soft fuzzy C *compact** space iff for any family of soft fuzzy sets $\{(\lambda_i^*, M_i^*)\}_{i\in J}$ in $SFCst$ with the property that $\sqcap_{j\in F}(\lambda_j^*, M_j^*) \neq (0, \phi)$ for any finite subset F of J, we have $\sqcap_{i\in J}(\lambda_i^*, M_i^*) \neq (0, \phi)$.

Property 4.4. The soft fuzzy C space $(\gamma(X), SFCst)$ is soft fuzzy C *compact**.

Proof. Let $\mathcal{F}$ be a soft fuzzy $SFCst$-prefilter on $\gamma(X)$. Let $(\nu, M) \in \mathcal{F}$. Since $(\nu, M) \in SFCst$, there is an index family $J_{(\nu,M)}$ such that $(\nu, M) = \sqcap_{j\in J_{(\nu,M)}}(\mu_j^*, N_j^*)$ for $(\mu_j, N_j) \in T'$. Since $(\nu, M) \sqsubseteq (\mu_j^*, N_j^*)$ for each $j \in J_{(\nu,M)}$. Therefore for each $(\nu, M) \in \mathcal{F}$ and $j \in J_{(\nu,M)}$, $(\mu_j^*, N_j^*) \in \mathcal{F}$. Consider the family of soft fuzzy closed sets in (X, T), $\mathcal{C} = \{(\mu, N) \in T'/(\mu^*, N^*) \in \mathcal{F}\}$. Since $(1, \gamma(X)) \in \mathcal{F}$, $(1, X) \in \mathcal{C}$. Thus $\mathcal{C}$ is nonempty.

(i) $(0^*, \phi^*) \notin \mathcal{F}$ implies $(0, \phi) \notin \mathcal{C}$.

(ii) If $(\mu_1, N_1),\ (\mu_2, N_2) \in \mathcal{C}$, then $(\mu_1^*, N_1^*),\ (\mu_2^*, N_2^*) \in \mathcal{F}$.

By definition of soft fuzzy $(T_{\gamma(X)})'$-prefilter, $(\mu_1^*, N_1^*) \sqcap (\mu_2^*, N_2^*) \in \mathcal{F}$. Which implies $(\mu_1, N_1) \sqcap (\mu_2, N_2) \in \mathcal{C}$. Hence $\mathcal{C}$ is a soft fuzzy base for the soft fuzzy T'-prefilter on X.

Let $\mathcal{U}_0$ be a soft fuzzy T'-prefilter contains $\mathcal{C}$. Then for each $(\mu, N) \in \mathcal{C}$,

$$(\mu^*(\mathcal{U}_0), N^*) = \begin{cases} (0, \phi), & \text{if } \mathcal{U}_0 \neq \mathcal{U}^x,\ \forall x \in X and \mu \notin \mathcal{U}_0,\ N = \phi, \\ (1, \gamma(X)), & \text{if } \mathcal{U}_0 \neq \mathcal{U}^x,\ \forall x \in X and \mu \in \mathcal{U}_0,\ N = X, \\ (\mu(x), \mathcal{U}_0), & \text{if } \exists\ x \in X \ni \mathcal{U}_0 = \mathcal{U}^x,\ x \in N. \end{cases}$$

This implies

$$(\mu^*(\mathcal{U}_0), N^*) = \begin{cases} (1, \gamma(X)), & \text{if } \mathcal{U}_0 \neq \mathcal{U}^x,\ \forall x \in X \text{ and } N = X, \\ \mu(x), & \text{if } \exists\ x \in X \ni \mathcal{U}_0 = \mathcal{U}^x \text{ and } x \in X. \end{cases}$$

$$\begin{aligned} (\bigwedge_{\mu \in \mathcal{C}} \mu^*(\mathcal{U}_0), \bigcap_{N \in \mathcal{C}} N^*) &= (\bigwedge_{\mu^* \in \mathcal{F}} \mu^*(\mathcal{U}_0), \bigcap_{N^* \in \mathcal{F}} N^*) \\ &= \begin{cases} (1, \gamma(X)), & \text{if } \mathcal{U}_0 \neq \mathcal{U}^x,\ \forall x \in X \text{ and } N = X, \\ (\wedge_{\mu \in \mathcal{C}} \mu(x), \mathcal{U}_0), & \text{if } \exists\ x \in X \ni \mathcal{U}_0 = \mathcal{U}^x,\ x \in \cap_{N \in \mathcal{C}} N. \end{cases} \end{aligned}$$

Hence $\sqcap_{(\mu,N)\in\mathcal{C}}(\mu^*(\mathcal{U}_0), N^*) \neq (0, \phi)$.

Now,

$$\begin{aligned} \sqcap_{(\nu,M)\in\mathcal{F}}(\nu, M) &= \sqcap_{(\nu,M)\in\mathcal{F}}(\sqcap_{j\in J_{(\nu,M)}}(\mu_j^*, N_j^*) \\ &= \sqcap_{(\mu,N)\in\mathcal{C}}(\mu^*, N^*). \end{aligned}$$
$$\Rightarrow \sqcap_{(\nu,M)\in\mathcal{F}}(\nu, M) \neq (0, \phi).$$

Thus $(\gamma(X), SFCst)$ is a soft fuzzy C *compact** space.

Property 4.5. Let (X, T) be a soft fuzzy topological space, F^* space. Suppose T' is a soft fuzzy normal family. Under these conditions $(\gamma(X), SFCst)$ is a soft fuzzy structure compactification of (X, T).

Proof. Proof the Property is obtained from Property 4.1 to Property 4.4.

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On generalized statistical convergence in random 2-normed spaces

Bipan Hazarika

Department of Mathematics, Rajiv Gandhi University, Rono Hills,
Doimukh, 791112, Arunachal Pradesh, India
E-mail: bh_rgu@yahoo.co.in

Abstract Recently in [20], Mursaleen introduced the concepts of statistical convergence in random 2-normed spaces. In this paper, we define and study the notion of λ-statistical convergence and λ-statistical Cauchy sequences in random 2-normed spaces, where $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1$, $\lambda_1 = 1$ and prove some interesting theorems.

Keywords Statistical convergence, λ-statistical convergence, t-norm, 2-norm, random 2-nor -med space.

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§1. Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [4] and Schoenberg [30] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view. For example, statistical convergence has been investigated and linked with summability theory by (Fridy [6], Šalát [29], Leindler [14], Mursaleen and Edely [25], Mursaleen and Alotaibi [21]), topological groups (Çakalli [2]), topological spaces (Maio and Kočinac [16]), measure theory (Millar [18], Connor, Swardson [3]), locally convex spaces (Maddox [15]). In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions [3].

The notion of statistical convergence depends on the (natural or asymptotic) density of

subsets of $\mathbf{N}$. A subset of $\mathbf{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k)$$

exists.

Definition 1.1. A sequence $x = (x_k)$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write S-$\lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

Karakus [11] studied the concept of statistical convergence in probabilistic normed spaces. The theory of probabilistic normed spaces was initiated and developed in [1], [17], [31], [32] and further it was extended to random/probabilistic 2-normed spaces [8] by using the concept of 2-norm [7].

Mursaleen [19] introduced the λ-statistical convergence for real sequences as follows:

Let $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{m+1} \leq \lambda_m + 1, \ \lambda_1 = 1.$$

The collection of such sequence λ will be denoted by Δ.

Let $K \subseteq \mathbf{N}$ be a set of positive integers. Then

$$\delta_\lambda(K) = \lim_m \frac{1}{\lambda_m} |\{m - \lambda_m + 1 \leq j \leq m : j \in K\}|$$

is said to be λ-density of K. In case $\lambda_m = m$, then λ-density reduces to natural density, so S_λ is same as S. Also, since $\left(\frac{\lambda_m}{m}\right) \leq 1$, $\delta(K) \leq \delta_\lambda(K)$, for every $K \subseteq \mathbf{N}$.

Definition 1.2.[19] A sequence $x = (x_k)$ is said to be λ-satistically convergent or S_λ-convergent to ℓ if for every $\varepsilon > 0$, the set $\{k \in I_m : |x_k - \ell| \geq \varepsilon\}$ has λ-density zero. In this case we write S_λ-$\lim x = \ell$ or $x_k \to \ell(S_\lambda)$ and

$$S_\lambda = \{x = (x_k) : \exists\ \ell \in \mathbf{R},\ S_\lambda\text{-}\lim x = \ell\}.$$

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed spaces [33] and intutionistic fuzzy normed spaces [12],[13],[26],[27],[28]. Further details on generalization of statistical convergence can be found in [22],[23],[24].

§2. Preliminaries

Definition 2.1. A function $f : \mathbf{R} \to \mathbf{R}_0^+$ is called a distribution function if it is a non-decreasing and left continuous with $\inf_{t\in\mathbf{R}} f(t) = 0$ and $\sup_{t\in\mathbf{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$. If $a \in \mathbf{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a, \\ 0, & \text{if } t \leq a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

A t-norm is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a,\ b,\ c \in [0,1]$. A triangle function τ is a binary operation on D^+, which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

In [7], Gähler introduced the following concept of 2-normed space.

Definition 2.2. Let X be a real vector space of dimension $d > 1$ (d may be infinite). A real-valued function $||.,.||$ from X^2 into $\mathbf{R}$ satisfying the following conditions:

(i) $||x_1, x_2|| = 0$ if and only if $x_1,\ x_2$ are linearly dependent,

(ii) $||x_1, x_2||$ is invariant under permutation,

(iii) $||\alpha x_1, x_2|| = |\alpha| ||x_1, x_2||$, for any $\alpha \in \mathbf{R}$,

(iv) $||x + \overline{x}, x_2|| \leq ||x, x_2|| + ||\overline{x}, x_2||$

is called an 2-norm on X and the pair $(X, ||.,.||)$ is called an 2-normed space.

A trivial example of an 2-normed space is $X = \mathbf{R}^2$, equipped with the Euclidean 2-norm $||x_1, x_2||_E$ = the volume of the parallellogram spanned by the vectors $x_1,\ x_2$ which may be given expicitly by the formula

$$||x_1, x_2||_E = |det(x_{ij})| = abs\left(\det(< x_i, x_j >)\right),$$

where $x_i = (x_{i1}, x_{i2}) \in \mathbf{R}^2$ for each $i = 1,\ 2$.

Recently, Goleţ [8] used the idea of 2-normed space to define the random 2-normed space.

Definition 2.3. Let X be a linear space of dimension $d > 1$ (d may be infinite), τ a triangle and $\mathcal{F} : X \times X \to D^+$. Then $\mathcal{F}$ is called a probabilistic 2-norm and $(X, \mathcal{F}, \tau)$ a probabilistic 2-normed space if the following conditions are satisfied:

$(P2N_1)$ $\mathcal{F}(x, y; t) = H_0(t)$ if x and y are linearly dependent, where $\mathcal{F}(x, y; t)$ denotes the value of $\mathcal{F}(x, y)$ at $t \in \mathbf{R}$,

$(P2N_2)$ $\mathcal{F}(x, y; t) \neq H_0(t)$ if x and y are linearly independent,

$(P2N_3)$ $\mathcal{F}(x, y; t) = \mathcal{F}(y, x; t)$, for all $x,\ y \in X$,

$(P2N_4)$ $\mathcal{F}(\alpha x, y; t) = \mathcal{F}(x, y; \frac{t}{|\alpha|})$, for every $t > 0,\ \alpha \neq 0$ and $x,\ y \in X$,

$(P2N_5)$ $\mathcal{F}(x + y, z; t) \geq \tau\left(\mathcal{F}(x, z; t), \mathcal{F}(y, z; t)\right)$, whenever $x,\ y,\ z \in X$.

If $(P2N_5)$ is replaced by $(P2N_6)$ $\mathcal{F}(x + y, z; t_1 + t_2) \geq \mathcal{F}(x, z; t_1) * \mathcal{F}(y, z; t_2)$, for all $x,\ y,\ z \in X$ and $t_1,\ t_2 \in \mathbf{R}_0^+$; then $(X, \mathcal{F}, *)$ is called a random 2-normed space (for short, $R2NS$).

Remark 2.1. Every 2-normed space $(X, ||.,.||)$ can be made a random 2-normed space in a natural way, by setting

(i) $\mathcal{F}(x, y; t) = H_0(t - ||x, y||)$, for every $x,\ y \in X,\ t > 0$ and $a * b = \min\{a, b\},\ a,\ b \in [0,1]$,

(ii) $\mathcal{F}(x, y; t) = \frac{t}{t + ||x, y||}$, for every $x,\ y \in X,\ t > 0$ and $a * b = ab,\ a,\ b \in [0,1]$.

Definition 2.4. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be convergent (or $\mathcal{F}$-convergent) to $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0,\ \theta \in (0,1)$ there exists an positive integer n_0 such that $\mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta$, whenever $k \geq n_0$ and for non zero $z \in X$. In this case we write $\mathcal{F}\text{-}\lim_k x_k = \ell$ and ℓ is called the $\mathcal{F}$-limit of $x = (x_k)$.

Definition 2.5. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be Cauchy with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\theta \in (0,1)$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $\mathcal{F}(x_k - x_m, z; \varepsilon) < 1 - \theta$, whenever $k,\ m \geq n_0$ and for non zero $z \in X$.

In [9], Gürdal and Pehlivan studied statistical convergence in 2-normed spaces and in 2-Banach spaces in [10]. In fact, Mursaleen [20] studied the concept of statistical convergence of sequences in random 2-normed spaces.

Definition 2.6.[20] A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be statistically-convergent or S^{R2N}-convergent to some $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\theta \in (0,1)$ and for non zero $z \in X$ such that

$$\delta\left(\{k \in \mathbf{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}\right) = 0.$$

In other words we can write the sequence (x_k) statistical converges to ℓ in random 2-normed space $(X, \mathcal{F}, *)$ if

$$\lim_{m\to\infty} \frac{1}{m} |\{k \leq m : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}| = 0,$$

or equivalently

$$\delta\left(\{k \in \mathbf{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta\}\right) = 1,$$

i.e.,

$$S\text{-}\lim_{k\to\infty} \mathcal{F}(x_k - \ell, z; \varepsilon) = 1.$$

In this case we write $S^{R2N} - \lim x = \ell$ and ℓ is called the S^{R2N}-limit of x. Let $S^{R2N}(X)$ denotes the set of all statistically convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

In this paper we define and study λ-statistical convergence in random 2-normed spaces which is quite a new and interesting idea to work with. We show that some properties of λ-statistical convergence of real numbers also hold for sequences in random 2-normed spaces. We find some relations related to λ-statistical convergent sequences in random 2-normed spaces. Also we find out the relation between λ-statistical convergent and λ-statistical Cauchy sequences in this spaces.

§3. λ-statistical convergence in random 2-normed spaces

In this section we define λ-statistically convergent sequence in random 2-normed $(X, \mathcal{F}, *)$. Also we obtained some basic properties of this notion in random 2-normed spaces.

Definition 3.1. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be λ-satistically convergent or S_λ-convergent to $\ell \in X$ with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\theta \in (0,1)$ and for non zero $z \in X$ such that

$$\delta_\lambda(\{k \in I_m : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}) = 0.$$

In other ways we write

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}| = 0,$$

or equivalently

$$\delta_\lambda\left(\{k \in I_m : \mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta\}\right) = 1,$$

i.e.,

$$S_\lambda\text{-}\lim_{k\to\infty} \mathcal{F}(x_k - \ell, z; \varepsilon) = 1.$$

In this case we write S_λ^{R2N}-$\lim x = \ell$ or $x_k \to \ell(S_\lambda^{R2N})$ and $S_\lambda^{R2N}(X) = \{x = (x_k) : \exists\ \ell \in \mathbf{R}, S_\lambda^{R2N}\text{-}\lim x = \ell\}$. Let $S_\lambda^{R2N}(X)$ denotes the set of all λ-statistically convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

Definition 3.2. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be λ-statistical Cauchy with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\theta \in (0, 1)$ and for non zero $z \in X$ there exists a positive integer $n = n(\varepsilon)$ such that for all $k,\ s \geq n$,

$$\delta_\lambda(\{k \in I_m : \mathcal{F}(x_k - x_s, z; \varepsilon) \leq 1 - \theta\}) = 0,$$

or equivalently

$$\delta_\lambda(\{k \in I_m : \mathcal{F}(x_k - x_s, z; \varepsilon) > 1 - \theta\}) = 1.$$

Definition 3.1, immediately implies the following Lemma.

Lemma 3.1. Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\lambda = (\lambda_n) \in \Delta$. If $x = (x_k)$ is a sequence in X, then for every $\varepsilon > 0$, $\theta \in (0, 1)$ and for non zero $z \in X$, then the following statements are equivalent:

(i) S_λ^{R2N}-$\lim_{k\to\infty} x_k = \ell$,

(ii) $\delta_\lambda(\{k \in I_m : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}) = 0$,

(iii) $\delta_\lambda\left(\{k \in I_m : \mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta\}\right) = 1$,

(iv) S_λ-$\lim_{k\to\infty} \mathcal{F}(x_k - \ell, z; \varepsilon) = 1$.

Theorem 3.1. Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\lambda = (\lambda_n) \in \Delta$. If $x = (x_k)$ is a sequence in X such that S_λ^{R2N}-$\lim x_k = \ell$ exists, then it is unique.

Proof. Suppose that there exist elements $\ell_1,\ \ell_2\ (\ell_1 \neq \ell_2)$ in X such that

$$S_\lambda^{R2N}\text{-}\lim_{k\to\infty} x_k = \ell_1; \quad S_\lambda^{R2N}\text{-}\lim_{k\to\infty} x_k = \ell_2.$$

Let $\varepsilon > 0$ be given. Choose $r > 0$ such that

$$(1 - r) * (1 - r) > 1 - \varepsilon. \tag{1}$$

Then, for any $t > 0$ and for non zero $z \in X$ we define

$$K_1(r, t) = \left\{k \in I_m : \mathcal{F}\left(x_k - \ell_1, z; \frac{t}{2}\right) \leq 1 - r\right\},$$

$$K_2(r, t) = \left\{k \in I_m : \mathcal{F}\left(x_k - \ell_2, z; \frac{t}{2}\right) \leq 1 - r\right\}.$$

Since S_λ^{R2N}-$\lim_{k\to\infty} x_k = \ell_1$ and S_λ^{R2N}-$\lim_{k\to\infty} x_k = \ell_2$, we have $\delta_\lambda(K_1(r, t)) = 0$ and $\delta_\lambda(K_2(r, t)) = 0$ for all $t > 0$.

Now let $K(r, t) = K_1(r, t) \cup K_2(r, t)$, then it is easy to observe that $\delta_\lambda(K(r, t)) = 0$. But we have $\delta_\lambda(K^c(r, t)) = 1$.

Now if $k \in K^c(r, t)$, then we have

$$\mathcal{F}(\ell_1 - \ell_2, z; t) \geq \mathcal{F}\left(x_k - \ell_1, z; \frac{t}{2}\right) * \mathcal{F}\left(x_k - \ell_2, z; \frac{t}{2}\right) > (1 - r) * (1 - r).$$

It follows by (1) that

$$\mathcal{F}(\ell_1 - \ell_2, z; t) > (1 - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{F}(\ell_1 - \ell_2, z; t) = 1$ for all $t > 0$ and non zero $z \in X$. Hence $\ell_1 = \ell_2$.

Next theorem gives the algebraic characterization of λ-statistical convergence on random 2-normed spaces.

Theorem 3.2. Let $(X, \mathcal{F}, *)$ be a random 2-normed space, $\lambda = (\lambda_n) \in \Delta$ and $x = (x_k)$ and $y = (y_k)$ be two sequences in X.

(i) If S_λ^{R2N}-$\lim x_k = \ell$ and $c(\neq 0) \in \mathbf{R}$, then S_λ^{R2N}-$\lim cx_k = c\ell$.

(ii) If S_λ^{R2N}-$\lim x_k = \ell_1$ and S_λ^{R2N}-$\lim y_k = \ell_2$, then S_λ^{R2N}-$\lim(x_k + y_k) = \ell_1 + \ell_2$.

Proof of the theorem is straightforward, thus omitted.

Theorem 3.3. Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\lambda = (\lambda_n) \in \Delta$. If $x = (x_k)$ be a sequence in X such that $\mathcal{F}$-$\lim x_k = \ell$ then S_λ^{R2N}-$\lim x_k = \ell$.

Proof. Let $\mathcal{F}$-$\lim x_k = \ell$. Then for every $\varepsilon > 0,\ t > 0$ and non zero $z \in X$, there is a positive integer n_0 such that

$$\mathcal{F}(x_k - \ell, z; t) > 1 - \varepsilon,$$

for all $k \geq n_0$. Since the set

$$K(\varepsilon, t) = \{k \in I_m : \mathcal{F}(x_k - \ell, z; t) \leq 1 - \varepsilon\}$$

has at most finitely many terms. Also, since every finite subset of $\mathbf{N}$ has δ_λ-density zero and consequently we have $\delta_\lambda(K(\varepsilon, t)) = 0$. This shows that S_λ^{R2N}-$\lim x_k = \ell$.

Remark 3.1. The converse of the above theorem is not true in general. It follows from the following example.

Example 3.1. Let $X = \mathbf{R}^2$, with the 2-norm $||x, z|| = |x_1 z_2 - x_2 z_1|$, $x = (x_1, x_2),\ z = (z_1, z_2)$ and $a * b = ab$ for all $a,\ b \in [0, 1]$. Let $\mathcal{F}(x, y; t) = \frac{t}{t + ||x, y||}$, for all $x,\ z \in X,\ z_2 \neq 0$ and $t > 0$. Now we define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} (k, 0), & \text{if } m - [\sqrt{\lambda_m}] + 1 \leq k \leq m, \\ (0, 0), & \text{otherwise.} \end{cases}$$

Nor for every $0 < \varepsilon < 1$ and $t > 0$, write

$$\begin{aligned} K(\varepsilon, t) &= \{k \in I_m : \mathcal{F}(x_k - \ell, z; t) \leq 1 - \varepsilon\} \quad \ell = (0, 0), \\ &= \{k \in I_m : \frac{t}{t + |x_k|} \leq 1 - \varepsilon\} \\ &= \{k \in I_m : |x_k| \geq \frac{t\varepsilon}{1 - \varepsilon} > 0\} \\ &= \{k \in I_m : m - [\sqrt{\lambda_m}] + 1 \leq k \leq m\}, \end{aligned}$$

so we get

$$\frac{1}{\lambda_m}|K(\varepsilon, t)| \leq \frac{1}{\lambda_m}|\{k \in I_m : m - [\sqrt{\lambda_m}] + 1 \leq k \leq m\}| \leq \frac{[\sqrt{\lambda_m}]}{\lambda_m}.$$

Taking limit m approaches to ∞, we get

$$\delta(K(\varepsilon, t)) = \lim_{m \to \infty} \frac{1}{\lambda_m}|K(\varepsilon, t)| \leq \lim_{m \to \infty} \frac{[\sqrt{\lambda_m}]}{\lambda_m} = 0.$$

This shows that $x_k \to 0(S_\lambda^{R2N}(X))$.

On the other hand the sequence is not $\mathcal{F}$-convergent to zero as

$$\mathcal{F}(x_k - \ell, z; t) = \frac{t}{t + |x_k|} = \begin{cases} \frac{t}{t+kz_2}, & \text{if } m - [\sqrt{\lambda_m}] + 1 \leq k \leq m, \\ 1, & \text{otherwise.} \end{cases} \leq 1.$$

Theorem 3.4. Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\lambda = (\lambda_n) \in \Delta$. If $x = (x_k)$ be a sequence in X, then S_λ^{R2N}-$\lim x_k = \ell$ if and only if there exists a subset $K \subseteq \mathbf{N}$ such that $\delta_\lambda(K) = 1$ and $\mathcal{F}$-$\lim x_k = \ell$.

Proof. Suppose first that S_λ^{R2N}-$\lim x_k = \ell$. Then for any $t > 0$, $r = 1, 2, 3, \cdots$ and non zero $z \in X$, let

$$A(r, t) = \left\{ k \in I_m : \mathcal{F}(x_k - \ell, z; t) > 1 - \frac{1}{r} \right\}$$

and

$$K(r, t) = \left\{ k \in I_m : \mathcal{F}(x_k - \ell, z; t) \leq 1 - \frac{1}{r} \right\}.$$

Since S_λ^{R2N}-$\lim x_k = \ell$ it follows that

$$\delta_\lambda(K(r, t)) = 0.$$

Now for $t > 0$ and $r = 1, 2, 3, \cdots$, we observe that

$$A(r, t) \supset A(r + 1, t)$$

and

$$\delta_\lambda(A(r, t)) = 1. \tag{2}$$

Now we have to show that, for $k \in A(r, t)$, $\mathcal{F}$-$\lim x_k = \ell$. Suppose that for $k \in A(r, t)$, (x_k) not convergent to ℓ with respect to $\mathcal{F}$. Then there exists some $s > 0$ such that

$$\{k \in I_m : \mathcal{F}(x_k - \ell, z; t) \leq 1 - s\}$$

for infinitely many terms x_k. Let

$$A(s, t) = \{k \in I_m : \mathcal{F}(x_k - \ell, z; t) > 1 - s\}$$

and

$$s > \frac{1}{r}, \ r = 1, 2, 3, \cdots.$$

Then we have

$$\delta_\lambda(A(s, t)) = 0.$$

Furthermore, $A(r, t) \subset A(s, t)$ implies that $\delta_\lambda(A(r, t)) = 0$, which contradicts (2) as $\delta_\lambda(A(r, t)) = 1$. Hence $\mathcal{F}$-$\lim x_k = \ell$.

Conversely, suppose that there exists a subset $K \subseteq \mathbf{N}$ such that $\delta_\lambda(K) = 1$ and $\mathcal{F}$-$\lim x_k = \ell$.

Then for every $\varepsilon > 0$, $t > 0$ and non zero $z \in X$, we can find out a positive integer n such that

$$\mathcal{F}(x_k - \ell, z; t) > 1 - \varepsilon$$

for all $k \geq n$. If we take

$$K(\varepsilon, t) = \{k \in I_m : \mathcal{F}(x_k - \ell, z; t) \leq 1 - \varepsilon\},$$

then it is easy to see that

$$K(\varepsilon, t) \subset \mathbf{N} - \{n_{k+1}, n_{k+2}, \cdots\}$$

and consequently

$$\delta_\lambda(K(\varepsilon, t)) \leq 1 - 1.$$

Hence S_λ^{R2N}-$\lim x_k = \ell$.

Finally, we establish the Cauchy convergence criteria in random 2-normed spaces.

Theorem 3.5. Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $\lambda = (\lambda_n) \in \Delta$. Then a sequence (x_k) in X is λ-statistically convergent if and only if it is λ-statistically Cauchy.

Proof. Let (x_k) be a λ-statistically convergent sequence in X. We assume that S_λ^{R2N}-$\lim x_k = \ell$. Let $\varepsilon > 0$ be given. Choose $r > 0$ such that (1) is satisfied. For $t > 0$ and for non zero $z \in X$ define

$$A(r,t) = \left\{k \in I_m : \mathcal{F}(x_k - \ell, z; \frac{t}{2}) \leq 1 - r\right\} \text{ and } A^c(r,t) = \left\{k \in I_m : \mathcal{F}(x_k - \ell, z; \frac{t}{2}) > 1 - r\right\}.$$

Since $S_\lambda^{R2N} - \lim x_k = \ell$ it follows that $\delta_\lambda(A(r,t)) = 0$ and consequently $\delta_\lambda(A^c(r,t)) = 1$. Let $p \in A^c(r,t)$. Then

$$\mathcal{F}(x_k - \ell, z; \frac{t}{2}) \leq 1 - r. \tag{3}$$

If we take

$$B(\varepsilon, t) = \{k \in I_m : \mathcal{F}(x_k - x_p, z; t) \leq 1 - \varepsilon\},$$

then to prove the result it is sufficient to prove that $B(\varepsilon, t) \subseteq A(r,t)$. Let $n \in B(\varepsilon, t)$, then for non zero $z \in X$,

$$\mathcal{F}(x_n - x_p, z; t) \leq 1 - \varepsilon. \tag{4}$$

If $\mathcal{F}(x_n - x_p, z; t) \leq 1 - \varepsilon$, then we have $\mathcal{F}(x_n - \ell, z; \frac{t}{2}) \leq 1 - r$ and therefore $n \in A(r,t)$. As otherwise i.e., if $\mathcal{F}(x_n - \ell, z; \frac{t}{2}) > 1 - r$ then by (1), (3) and (4) we get

$$\begin{aligned} 1 - \varepsilon \geq \mathcal{F}(x_n - x_p, z; t) &\geq \mathcal{F}(x_n - \ell, z; \frac{t}{2}) * \mathcal{F}(x_p - \ell, z; \frac{t}{2}) \\ &> (1 - r) * (1 - r) > (1 - \varepsilon), \end{aligned}$$

which is not possible. Thus $B(\varepsilon, t) \subset A(r,t)$. Since $\delta_\lambda(A(r,t)) = 0$, it follows that $\delta_\lambda(B(\varepsilon, t)) = 0$. This shows that (x_k) is λ-statistically Cauchy.

Conversely, suppose (x_k) is λ-statistically Cauchy but not λ-statistically convergent. Then there exists positive integer p and for non zero $z \in X$ such that if we take

$$A(\varepsilon, t) = \{k \in I_m : \mathcal{F}(x_n - x_p, z; t) \leq 1 - \varepsilon\},$$

then

$$\delta_\lambda(A(\varepsilon, t)) = 0$$

and consequently

$$\delta_\lambda(A^c(\varepsilon, t)) = 1. \tag{5}$$

Since

$$\mathcal{F}(x_n - x_p, z; t) \geq 2\mathcal{F}(x_k - \ell, z; \frac{t}{2}) > 1 - \varepsilon,$$

if $\mathcal{F}(x_k - \ell, z; \frac{t}{2}) > \frac{1-\varepsilon}{2}$, then we have

$$\delta_\lambda(\{k \in I_m : \mathcal{F}(x_n - x_p, z; t) > 1 - \varepsilon\}) = 0.$$

i.e., $\delta_\lambda(A^c(\varepsilon, t)) = 0$, which contradicts (5) as $\delta_\lambda(A^c(\varepsilon, t)) = 1$. Hence (x_k) is λ-statistically convergent.

Combining Theorem 3.4 and Theorem 3.5 we get the following corollary.

Corollary 3.1. Let $(X, \mathcal{F}, *)$ be a random 2-normed space and and $x = (x_k)$ be a sequence in X. Then the following statements are equivalent:

(i) x is λ-statistically convergent,

(ii) x is λ-statistically Cauchy,

(iii) There exists a subset $K \subseteq \mathbf{N}$ such that $\delta_\lambda(K) = 1$ and $\mathcal{F}\text{-}\lim x_k = \ell$.

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New operation compact spaces

(Dedicated to Prof. Dr. Takashi Noiri on his 68th birthday)

Sabir Hussain
Department of Mathematics, Yanbu University, P. O. Box,
31387, Yanbu Alsinaiyah, Saudi Arabia
E-mail: sabiriub@yahoo.com

Abstract In this paper, our main tool is the use of mapping $\beta : SO(X) \to P(X)$ from the semi-open set into power set $P(X)$ of the underlying set X, having the property of monotonicity and $A \subseteq \beta(A)$ for each semi-open sets A, where $\beta(A)$ denotes the value of A under β. Moreover, the operation β generalize the notions and characterizations defined and discussed using the operation α defined and discussed by Kasahara [6].

Keywords β-convergence, β-accumulation, β-continuous mapping, β-compact.

§1. Introduction

Topology is the branch of mathematics, whose concepts exist not only in all branches of mathematics, but also has the independent theoretic framework, background and broad applications in many real life problems. A lot of researchers working on the different structures of topological spaces. In [5], S. Kasahara proved that a T_1 space Y is compact if and only if for every topological space X, each mapping of X in Y with a closed graph is continuous. Then by a similar methods, Herrington and Long [2] proved that a Hausdroff space Y is H-closed if and only if for every topological space X, each mapping of X into Y with a strong closed graph is weakly continuous. It was shown by Joesph [4] that, in the results of Herrington [2], "mapping" can be replaced with "bijection". Moreover, using the characterizations of compactness mentioned above, Herrington and Long [3] obtained that a weakly Hausdroff space Y is nearly compact if and only if for every topological space X, each mapping of X into Y with a closed graph is almost continuous. Further Thompson [8] proved that a Hausdorff space Y is nearly compact if and only if for every topological space X, each mapping of X into Y with an r-closed graph is almost continuous. N. Levine [7] initiated the study of semi-open sets. A subset A of a space X is said to be a semi-open set [7], if there exists an open set O such that $O \subseteq A \subseteq cl(O)$. The set of all semi-open sets is denoted by $SO(X)$. A is semi-closed iff $X - A$ is semi-open in X. The intersection of all semi-closed sets containing A is called semi-closure [1] of A and is denoted by $scl(A)$. The concept of operation α on a topological spaces was introduced by S. Kasahara [6]. He showed that several known characterizations of compact spaces, nearly compact spaces and H-closed spaces are unified by generalizing the notion of compactness with the help of a certain operation of topology τ into power set of U_τ. Many reserachers worked on this operation α intoduced by Kasahara and a lot of material is available in the literature.

The purpose of this paper is the use of mapping $\beta : SO(X) \rightarrow P(X)$ from the semi-open set into power set $P(X)$ of the underlying set X as our main tool, having the property of monotonicity and $A \subseteq \beta(A)$ for each semi-open sets A, where $\beta(A)$ denotes the value of A under β. Moreover, the operation β generalize the notions and characterizations defined and discussed using the operation α defined and discussed by Kasahara [6].

§2. β-convergence and β-accumulation

We start with a mapping $\beta : SO(X) \rightarrow P(X)$ from the semi-open set into power set $P(X)$ of the underlying set X, having the property of monotonicity (i.e., $A \subseteq B$ implies $\beta(A) \subseteq \beta(B)$) and $A \subseteq \beta(A)$ for each semi-open sets A, where $\beta(A)$ denotes the value of A under β. The mapping β defined by $\beta(A) = A$ for each $A \in SO(X)$ is an operation on $SO(X)$ called the identity operation. The semi-closure operation of course defines an operation on $SO(X)$ and the composition *sint o scl* of the semi-closure operation with semi-interior operation *sint* is also an operation on $SO(X)$.

Definition 2.1. Let (X, τ) be a topological space and β be an operation on $SO(X)$. A filter base Γ in X, β-converges to $a \in X$, if for every semi open nbd V of a, there exists an $F \in \Gamma$ such that $F \subset V^{\beta}$.

Definition 2.2. Let (X, τ) be a topological space and β be an operation on $SO(X)$. A filter base Γ in X, β-accumulates to $a \in X$, if $F \cap V^{\beta} \neq \phi$, for every $F \in \Gamma$ and for every semi-open nbd V of a, where V^{β} denotes the value of V under β.

For the identity operation β, the convergence and accumulation contains in β-convergence and β-accumulate. Furthermore, α-convergence and α-accumulate in the sense of Kasahara [6] also contains in β-convergence and β-accumulates.

Theorem 2.3. Let (X, τ) be a topological space and β be an operation on $SO(X)$. Then the following hold:

(i) If a filterbase Γ in X, β-converges to $a \in X$, then Γ β-accumulates to a.

(ii) If a filterbase Γ in X is contained in a filterbase which β-accumulates to $a \in x$, then Γ β-accumulates to a.

(iii) If a maximal filterbase in X, β-accumulates to $a \in X$ then it β-converges to a.

Proof. The proof is obvious.

Note that regularity of operation α in the sense of S. Kasahara [6] follows from the monotonicity of operation β.

Theorem 2.4. Let (X, τ) be a topological space and β be an operation of $SO(X)$. If a filter base Γ in X, β-accumulates to $a \in X$, then there exists a filterbase Λ in X such that $\Gamma \subset \Lambda$ and Λ, β-converges to a.

Proof. It follows form the monotonicity of operation β that the set $\Lambda^{'} = \{F \cap V^{\beta} : F \in \Gamma$ and $a \in V \in SO(X)\}$ is a filterbase in X. Clearly the filterbase Λ generated by Λ', β-converges to a and $\Gamma \subset \Lambda$.

§3. β-continuous mappings

Let (X, τ), $(Y, \tau^{'})$ be two topological spaces and β be an operation on $SO(Y)$. A mapping f of X into Y is said to be β-continuous at $a \in X$, if for every semi-open nbd V of $f(a)$, there is a semi-open nbd U of a such that $f(U) \subset V^{\beta}$. Clearly f is β-continuous at $a \in X$ if and only if for every filterbase Γ in X converging to a, the filterbase $f(\Gamma)$ β-converges to $f(a)$. A mapping $f : X \to Y$ is β-continuous, if f is β-continuous at each point of X.

Theorem 3.1. Let (X, τ), $(Y, \tau^{'})$ be two topological spaces and β be an operation on $SO(Y)$. If a mapping f of X into Y is β-continuous, then $V \in SO(Y)$ and $V^{\beta} = V$ imply $f^{-1}(V) \in SO(X)$. The converse is true if $V^{\beta\beta} = V^{\beta} \in SO(Y)$ for all $V \in SO(Y)$.

Proof. Suppose $V \in SO(Y)$ and $V^{\beta} = V$. Then for each $x \in f^{-1}(G)$, we can find semi-open nbd U of x such that $f(U) \subset V^{\beta} = V$, which implies $U \subset f^{-1}(V)$. Therefore $f^{-1}(V) \in SO(X)$. Suppose now that $V^{\beta\beta} = V^{\beta} \in SO(Y)$ for all $V \in SO(Y)$ and let $a \in X$. Then for each semi-open nbd U of $f(a)$, the set $f^{-1}(U^{\beta}) \in SO(X)$, since $U^{\beta\beta} = U^{\beta} \in SO(Y)$. From $f(a) \in U \subset U^{\beta}$. It follows therefore that $f^{-1}(U^{\beta})$ is semi-open nbd of a. Hence the proof.

Definition 3.2. Let (X, τ), $(Y, \tau^{'})$ be two topological spaces and β be an operation on $SO(Y)$. The graph $G(f)$ of $f : X \to Y$ is β-closed if for each $(x, y) \in (X \times Y) \backslash G(f)$ there exist semi-open sets U and V containing x and y respectively such that $(U \times V^{\beta}) \cap G(f) = \phi$.

Now we characterize β-closed graph which also gives generalization of the Theorem 4 of [6].

Theorem 3.3. Let (X, τ), $(Y, \tau^{'})$ be two topological spaces, β be an operation on $SO(Y)$ and $f : X \to Y$ be a mapping from (X, τ) into $(Y, \tau^{'})$. The the following are equivalent:

(i) f has a β-closed graph.

(ii) Let $a \in X$ and there exists a filterbase Γ in X converges to a such that $f(\Gamma)$ β-accumulates to $b \in Y$, then $f(a) = b$.

(iii) Let $a \in X$ and there exists a filterbase Γ in X converges to a such that $f(\Gamma)$ β-converges to $b \in Y$, then $f(a) = b$.

Proof. (i)⇒(ii) Contrarily suppose that $f(a) \neq b$. Then since $G(f)$ is β-closed, there exists semi-open sets U and V such that $a \in U$, $b \in V$ and $(U \times V^{\beta}) \cap G(f) = \phi$. Hence we can find an $F \in \Gamma$ such that $F \subset U$ and $f(F) \cap V^{\beta} \neq \phi$. However this implies $f(x) \in V^{\beta}$ for some $x \in F$, which yields $(x, f(x)) \in U \times V^{\beta}$. A contradiction.

(ii)⇒(iii) This follows directly form Theorem 2.3.

(iii)⇒(i) Contrarily suppose that (i) is not true. Then there exists a $(a, b) \in (X \times Y) \backslash G(f)$ such that $(U \times V^{\beta}) \cap G(f) \neq \phi$ for every semi open nbd U of a and for every semi open nbd V of b. Since β is monotone follows that $\Gamma = \{U \cap f^{-1}(V^{\beta}) : a \in U \in SO(X) \text{ and } b \in V \in SO(Y)\}$ is a filterbase in X. Clearly Γ converges to a and $f(\Gamma)$ β-converges to b. Which implies that (iii) does not hold. This completes the proof.

Definition 3.4. Let (X, τ), $(Y, \tau^{'})$ be two topological spaces, β be an operation on $SO(Y)$, then $f : X \to Y$ is said to be β-subcontinuous, if for every convergent filterbase Γ in X, the filterbase $f(\Gamma)$ β-accumulates to some point of Y.

Note that, by Theorem 2.3(i), a β-continuous mapping is β-subcontinuous. For converse implication, we have the following theorem:

Theorem 3.5. Let (X, τ), $(Y, \tau^{'})$ be two topological spaces, β be an operation on $SO(Y)$. If f is β-subcontinuous mapping of X into Y with a β-closed graph, then f is β-continuous.

Proof. Contrarily suppose that f is not β-continuous. Then there exists a $a \in X$ and a filterbase Γ in X converges to a such that $f(\Gamma)$ does not β-converges to $f(a)$. Hence there exists a semi-open nbd V of $f(a)$ such that $f(F) \cap V^{\beta^c} \neq \phi$, where V^{β^c} denotes the complement of V^{β} in Y. Clearly, Γ is contained in the filter Λ generated by Λ', and so Λ converges to a. But $f(\Lambda)$ does not β-accumulate to $f(a)$. On the other hand, since f is β-contnuous, the filterbase $f(\Lambda)$ β-converges to some $b \in Y$. By Theorem 3.3, we have $f(a) = b$, which is absurd. This completes the proof.

§4. β-compact

Definition 4.1. Let (X, τ) be topological space and β be an operation on $SO(X)$. A subset A of X is said to be β-compact, if for every open cover C of K there exists a finite subfamily $\{G_1, G_2, \cdots, G_n\}$ of C such that $A \subset \bigcup_{i=1}^{n} G_i^{\beta}$.

Definition 4.2. Let (X, τ) be topological space and β be an operation on $SO(X)$. Then space X is said to be β-regular, if for each $x \in X$ and for each semi open nbd V of x, there exists a semi open nbd U of x such that $U^{\beta} \subset V$.

Clearly compact space (X, τ) is β-compact. For the converse, we prove the following.

Theorem 4.3. Let (X, τ) be topological space and β be an operation on $SO(X)$. If (X, τ) is β-regular space then (X, τ) is compact.

Proof. Let C be a semi open cover of X. Then β-regularity of X implies that the set D of all $V \in SO(X)$ such that $V^{\beta} \subset U$ for some $U \in D$ is a semi open cover of X. Hence $X = \bigcup_{i=1}^{n} V_i^{\beta}$ for some $V_1,\ V_2,\ \cdots,\ V_n \in D$. For each $i \in \{1, 2, \cdots, n\}$, there exists a $U_i \in D$ such that $V_i^{\beta} \subset U_i$. Therefore, we have $X = \bigcup_{i=1}^{n} G_i$. This is completes the proof.

Theorem 4.4. Let (X, τ) be topological space and β be an operation on $SO(X)$. Then the following are equivalent:

(i) (X, τ) is β-compact.

(ii) Each filterbase in X β-accumulates to some point of X.

(iii) Each maximal filterbase in X β-converges to some point of X.

Proof. (i)⇒(ii) Contrarily suppose that there exists a filterbase Γ in X which does not β-accumulate to any point of X. Then for each $x \in X$, there exist a $F_x \in \Gamma$ and semi open nbd V_x of x such that $F_x \cap V_x^{\beta} = \phi$. Since the family $\{V_x : x \in X\}$ is a semi open cover of X. Since X is β-compact then we have $X = \bigcup_{i=1}^{n} V_{x_i}^{\beta}$ for some $x_1,\ x_2,\ \cdots,\ x_n \in X$; but then $\bigcap_{i=1}^{n} F_{x_i} \neq \phi$ and consequently we have $F_{x_m} \cap V_{x_m}^{\beta} \neq \phi$, for some $m \in \{1, 2, \cdots, n\}$, a contradiction.

(ii)⇒(iii) Since each filterbase is contained in a maximal filterbase in X. This follows directly form (iii) of Theorem 2.3.

(iii)⇒(ii) This is obvious.

(ii)⇒(i) Contrarily suppose that (ii) holds and that there exists a semi open cover C of X such that $X \neq \bigcup_{i=1}^{n} G_i^{\beta}$ for any finite $\{G_1, G_2, \cdots, G_n\} \subset C$. Let Γ denote the set of all sets of the form $\bigcap_{i=1}^{n} G_i^{\beta^c}$ where $n \in Z^+$ and $G_i \subset C$ and $G_i^{\beta^c}$ denotes the complement of G_i^{β} in

X. then since Γ is filterbase in X, it β-accumulates to some $a \in x$. But then $a \notin G$ for some $G \in C$ and so $G^{\beta^c} \in \Gamma$ yields a contradiction $G^{\beta^c} \cap G \neq \phi$. This completes the proof.

Theorem 4.5. Let (X, τ), $(Y, \tau^{'})$ be two topological spaces and β be an operation on $SO(Y)$ and $f : X \to Y$ be a mapping from (X, τ) into $(Y, \tau^{'})$. If (X, τ) is compact and f is β-continuous, then $f(X)$ is β-compact.

Proof. Let C be a semi open cover of $f(X)$. Denote by D the set of all $U \in SO(X)$ such that $f(U) \subset V^{\beta}$ for some $V \in C$. Then by β-continuity of f D is semi open cover of X. Hence $X = \bigcup_{i=1}^{n} U_i$ for some $U_1,\ U_2,\ \cdots,\ U_n \in D$. For each $i \in \{1, 2, \cdots, n\}$, we can find a $V_i \in C$ such that $f(U_i) \subset V_i^{\beta}$. Therefore we have $f(X) \subset \bigcup_{i=1}^{n} V_i^{\beta}$. Hence the proof.

Theorem 4.6. If $(Y, \tau^{'})$ is a β-compact space for some operation β on $SO(Y)$, then every mapping f of any topological space (X, τ) into Y is β-subcontinuous.

Proof. Let Γ be a convergent filterbase in X. Then by Theorem 4.4, the filterbase $f(\Gamma)$ β-accumulates to some point in Y. Thus f is β-subcontinuous.

Theorem 4.7. Let f be a mapping of a topological space (X, τ) into another topological space $(Y, \tau^{'})$ and β be an operation on $SO(Y)$. If $(Y, \tau^{'})$ is β-compact and f has a β-closed graph, then f is β-continuous.

Proof. By Theorem 4.6, f is β-subcontinuous and hence it is β-continuous by Theorem 3.5. Hence the proof.

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A common fixed point theorem in Menger space

S. Chauhan[†], B. D. Pant[‡] and N. Dhiman[♯]

† R. H. Government Postgraduate College,Kashipur, 244713,
U. S. Nagar, Uttarakhand, India
‡ Government Degree College, Champawat, 262523,
Uttarakhand, India
♯ Department of Mathematics, Graphic Era University,
Dehradun, Uttarakhand, India
E-mails: sun.gkv@gmail.com badridatt.pant@gmail.com

Abstract Al-Thagafi and Shahzad [4] introduced the notion of occasionally weakly compatible mappings which is more general than the notion of weakly compatible mappings. The aim of the present paper is to prove a common fixed point theorem for occasionally weakly compatible mappings in Menger space satisfying a new contractive type condition. Our result never requires the completeness of the whole space (or any subspace), continuity of the involved mappings and containment of ranges amongst involved mappings. An example is also derived which demonstrates the validity of our main result.

Keywords Triangle norm, menger space, weakly compatible mappings, occasionally weakly compatible mappings, fixed point.

§1. Introduction

Karl Menger [16] introduced the notion of probabilistic metric space (shortly, PM-space), which is a generalization of the metric space. The study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [21,22].

In 1986, Jungck [11] introduced the notion of compatible mappings in metric spaces. Mishra [17] extended the notion of compatibility to probabilistic metric spaces. This condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [12,13]. The concept of weakly compatible mappings is most general as each pair of compatible mappings is weakly compatible but the reverse is not true. In 2002, Aamri and El-Moutawakil [1] introduced the well known concept (E.A) property and generalized the concept of non-compatible mappings. The results obtained in the metric fixed point theory by using the notion of non-compatible mappings or (E.A) property are very interesting. Lastly, Al-Thagafi and Shahzad [4] introduced the notion of occasionally weakly compatible mappings which is more general than the concept of weakly compatible mappings. Several interesting and elegant results have been obtained by various authors in this direction [2, 3, 5-10, 14, 15, 18-20, 25].

In the present paper, we prove a common fixed point theorem for occasionally weakly compatible mappings in Menger space satisfying a new contractive type condition. Our result

improves many known results in the sense that the conditions such as completeness of the whole space (or any subspace), continuity of the involved mappings and containment of ranges amongst involved mappings are completely relaxed.

§2. Preliminaries

Definition 2.1.[22] A triangular norm T (shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a,\ b,\ c,\ d \in [0,1]$ and the following conditions are satisfied:

(i) $\mathrm{T}(a,1) = a$,

(ii) $\mathrm{T}(a,b) = \mathrm{T}(b,a)$,

(iii) $\mathrm{T}(a,b) \leq \mathrm{T}(c,d)$ for $a \leq c$, $b \leq d$,

(iv) $\mathrm{T}\left(\mathrm{T}(a,b),c\right) = \mathrm{T}\left(a,\mathrm{T}(b,c)\right)$.

Examples of t-norms are:

$$\mathrm{T}_M(a,b) = \min\{a,b\},\ \mathrm{T}_P(a,b) = a.b \text{ and } \mathrm{T}_L(a,b) = \max\{a+b-1,0\}.$$

Now t-norms are recursively defined by $\mathrm{T}^1 = \mathrm{T}$ and

$$\mathrm{T}^n(x_1,\cdots,x_{n+1}) = \mathrm{T}(\mathrm{T}^{n-1}(x_1,\cdots,x_n),x_{n+1}).$$

Definition 2.2.[22] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.3.[22] A PM-space is an ordered pair $(X,\mathcal{F})$, where X is a non-empty set and $\mathcal{F}$ is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$. The function $F_{x,y}$ are assumed to satisfy the following conditions:

(i) $F_{x,y}(0) = 0$,

(ii) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $F_{x,y}(t) = F_{y,x}(t)$,

(iv) $F_{x,z}(t) = 1, F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t+s) = 1$, for all $x,\ y,\ z \in X$ and $t,\ s > 0$.

The ordered triple $(X,\mathcal{F},\mathrm{T})$ is called a Menger space if $(X,\mathcal{F})$ is a PM-space, T is a t-norm and the following inequality holds:

$$F_{x,y}(t+s) \geq \mathrm{T}(F_{x,z}(t),F_{z,y}(s)),$$

for all $x,\ y,\ z \in X$ and $t,\ s > 0$. Every metric space (X,d) can be realized as a PM-space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x,y))$ for all $x,\ y \in X$.

Definition 2.4.[22] Let $(X,\mathcal{F},\mathrm{T})$ be a Menger space with continuous t-norm T.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point x in X if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N(\epsilon, \lambda)$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n,\ m \geq N(\epsilon, \lambda)$.

(iii) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5.[17] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \mathrm{T})$ are said to be compatible if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 2.6.[14] Let A and S be two self mappings of non-empty set X. A point $x \in X$ is called a coincidence point of A and S if and only if $Ax = Sx$. In this case $w = Ax = Sx$ is called a point of coincidence of A and S.

Definition 2.7.[12] Two self mappings A and S of a non-empty set X are said to be weakly compatible if they commute at their coincidence points, that is, if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Remark 2.1.[24] If self mappings A and S of a Menger space $(X, \mathcal{F}, \mathrm{T})$ are compatible then they are weakly compatible but converse is not true.

Definition 2.8.[14] Two self mappings A and S of a non-empty set X are occasionally weakly compatible if and only if there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute.

Lemma 2.1.[14] Let X be a non-empty set, A and S are occasionally weakly compatible self mappings of X. If A and S have a unique point of coincidence, $w = Ax = Sx$, then w is the unique common fixed point of A and S.

Proposition 2.1.[23] A binary operation $\mathrm{T} : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm and satisfies the condition

$$\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty} \left(1 - \alpha^i(t)\right) = 1,$$

where $\alpha : \mathbb{R}^+ \to (0,1)$. It is easy to see that this condition implies $\lim_{n\to\infty} \alpha^n(t) = 0$.

§3. Result

Theorem 3.1. Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger space with continuous t-norm T. Further, let $A,\ L,\ M$ and S be self-mappings of X and the pairs (L, A) and (M, S) be each occasionally weakly compatible satisfying condition:

$$F_{Lx,My}(t) \geq 1 - \alpha(t) \left(1 - F_{Ax,Sy}(t)\right), \tag{1}$$

for all $x,\ y \in X$ and every $t > 0$, where $\alpha : \mathbb{R}^+ \to (0,1)$ is a monotonic increasing function.

If

$$\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty} \left(1 - \alpha^i(t)\right) = 1.$$

Then there exists a unique point $w \in X$ such that $Aw = Lw = w$ and a unique point $z \in X$ such that $Mz = Sz = z$. Moreover, $z = w$, so that there is a unique common fixed point of $A,\ L,\ M$ and S.

Proof. Since the pairs (L, A) and (M, S) are occasionally weakly compatible, there exist points $u,\ v \in X$ such that $Lu = Au$, $LAu = ALu$ and $Mv = Sv,\ MSv = SMv$. Now we show that $Lu = Mv$. By putting $x = u$ and $y = v$ in inequality (1), we get

$$\begin{aligned} F_{Lu,Mv}(t) &\geq 1 - \alpha(t)(1 - F_{Au,Sv}(t)) \\ &= 1 - \alpha(t)(1 - F_{Lu,Mv}(t)) \\ &\geq 1 - \alpha(t)\{1 - (1 - \alpha(t)(1 - F_{Lu,Mv}(t)))\} \\ &= 1 - \alpha^2(t)(1 - F_{Lu,Mv}(t)) \\ &\geq \ldots \geq 1 - \alpha^n(t)(1 - F_{Lu,Mv}(t)) \to 1. \end{aligned}$$

Thus, we have $Lu = Mv$. Therefore, $Lu = Au = Mv = Sv$. Moreover, if there is another point z such that $Lz = Az$. Then using the inequality (1), it follows that $Lz = Az = Mv = Sv$, or $Lu = Lz$. Hence $w = Lu = Au$ is the unique point of coincidence of L and A. By Lemma 2.1, w is the unique common fixed point of L and A. Similarly, there is a unique point $z \in X$ such that $z = Mz = Sz$. Suppose that $w \neq z$ and taking $x = u$ and $y = z$ in inequality (1), we get

$$\begin{aligned} F_{Lu,Mz}(t) &\geq 1 - \alpha(t)(1 - F_{Au,Sz}(t)), \\ F_{w,z}(t) &\geq 1 - \alpha(t)(1 - F_{w,z}(t)) \\ &\geq 1 - \alpha(t)\{1 - (1 - \alpha(t)(1 - F_{w,z}(t)))\} \\ &= 1 - \alpha^2(t)(1 - F_{w,z}(t)) \\ &\geq \ldots \geq 1 - \alpha^n(t)(1 - F_{w,z}(t)) \to 1. \end{aligned}$$

Thus, we have, $w = z$. So w is the unique common fixed point of the mappings $A,\ L,\ M$ and S.

Now, we give an example which illustrates Theorem 3.1.

Example 3.1. Let $X = [0, 1]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x,\ y \in X$. Clearly $(X, \mathcal{F}, \mathrm{T})$ be a Menger space, where T is a continuous t-norm. Define the self mappings $A,\ L,\ M$ and S by

$$L(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \quad A(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

$$M(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \quad S(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{x}{4}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Then $A,\ L,\ M$ and S satisfy all the conditions of Theorem 3.1 with respect to the distribution function $F_{x,y}$.

First, we have

$$L(\frac{1}{2}) = \frac{1}{2} = A(\frac{1}{2}) \text{ and } LA(\frac{1}{2}) = \frac{1}{2} = AL(\frac{1}{2})$$

and

$$M(\frac{1}{2}) = \frac{1}{2} = S(\frac{1}{2}) \text{ and } MS(\frac{1}{2}) = \frac{1}{2} = SM(\frac{1}{2}),$$

that is, L and A as well as M and S are occasionally weakly compatible. Also $\frac{1}{2}$ is the unique common fixed point of $A,\ L,\ M$ and S. On the other hand, it is clear to see that the mappings $A,\ L,\ M$ and S are discontinuous at $\frac{1}{2}$. Moreover, this example does not hold any condition on the containment of ranges amongst involved mappings.

By choosing $A,\ L,\ M$ and S suitably, we can deduce corollaries for a pair as well as for a triod of self mappings. The details of two possible corollaries for a triod of mappings are not included. As a sample, we outline the following natural result for a pair of self mappings.

Corollary 3.1. Let $(X, \mathcal{F}, \mathrm{T})$ be a Menger space with continuous t-norm T. Further, let A and L be self mappings of X and the pair (L, A) is occasionally weakly compatible satisfying condition:

$$F_{Lx,Ly}(t) \geq 1 - \alpha(t)\left(1 - F_{Ax,Ay}(t)\right), \tag{2}$$

for all $x,\ y \in X$ and every $t > 0$, where $\alpha : \mathbb{R}^+ \to (0,1)$ is a monotonic increasing function.

If

$$\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty}\left(1 - \alpha^i(t)\right) = 1.$$

Then A and L have a unique common fixed point in X.

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The strongly generalized double difference χ sequence spaces defined by a modulus

N. Subramanian† and U. K. Misra‡

† Department of Mathematics,SASTRA University,
Thanjavur, 613401, India.
‡ Department of Mathematics, Berhampur University,
Berhampur, 760007, Odissa, India.
E-mail: nsmaths@yahoo.com umakanta_misra@yahoo.com

Abstract In this paper we introduce the strongly generalized difference

$$V_{2\chi^2}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$$
$$= \left\{x=(x_{mn})\in w^2 : \lim_{r,s\to\infty}\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}=0\right\},$$

$$V_{2\Lambda^2}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$$
$$= \left\{x=(x_{mn})\in w^2 : \sup_{r,s}\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left(\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}<\infty\right\},$$

where f is a modulus function and $A_i = a_{i(k,\ell)}^{i(mn)}$ is a nonnegative four dimensional matrix of complex numbers and $p_{i(mn)}$ is a sequence of positive real numbers. We also give natural relationship between strongly generalized difference $V_{2\chi^2}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$-summable sequences with respect to f. We examine some topological properties of $V_{2\chi^2}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$ spaces and investigate some inclusion relations between these spaces.

Keywords De la Vallee-Poussin mean, difference sequence, gai sequence, analytic sequence, modulus function, double sequences.
2000 Mathematics subject classification: 40A05, 40C05, 40D05.

§1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m,\ n\in\mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [6], Moricz [10], Moricz and Rhoades [11], Basarir and Solankan [2],

Tripathy [18], Colak and Turkmenoglu [5], Turkmenoglu [20] and many others.

Quite recently, in her PhD thesis, Zelter [24] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [25] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [26] and Mursaleen and Edely [27] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [28] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [29] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [30] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [32], Maddox [33] and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [9] as an extension of the definition of strongly Cesàro summable sequences. Connor [34] further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. In [35] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [36]-[40] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary (single) sequence spaces to multiply sequence spaces. A sequence $x = \left(x_{i(mn)}\right)$ is said to be strongly (V_2, λ_2) summable to zero if $t_{rs}(|x|) \to 0$ as $r, s \to \infty$. Let $A = \left(a^{i(mn)}_{i(k,\ell)}\right)$ be an infinite four dimensional matrix of complex numbers. We write $Ax = (A_i(x))_{i=1}^{\infty}$ if $A_i(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(a^{i(mn)}_{i(k,\ell)}\right) x_{mn}$ converges for each $i \in \mathbb{N}$.

Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 < p_{mn} < supp_{mn} = G$ and let $D = max\left(1, 2^{G-1}\right)$. Then for $a_{mn},\ b_{mn} \in \mathbb{C}$, the set of complex numbers for all $m,\ n \in \mathbb{N}$ we have

$$|a_{mn} + b_{mn}|^{\frac{1}{m+n}} \le D\left\{|a_{mn}|^{\frac{1}{m+n}} + |b_{mn}|^{\frac{1}{m+n}}\right\}. \qquad (1)$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called

double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [31] as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0 and ℓ_∞ denote the classes of all, convergent, null and bounded sclar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1} |\Delta x_k|.$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$, $\Delta^m x_{mn} = \Delta\Delta^{m-1}x_{mn}$ for all $m, n \in \mathbb{N}$, where $\Delta^m x_{mn} = \Delta^{m-1}x_{mn} - \Delta^{m-1}x_{mn+1} - \Delta^{m-1}x_{m+1,n} + \Delta^{m-1}x_{m+1,n+1}$, for all $m, n \in \mathbb{N}$.

Definition 1.1. A modulus function was introduced by Nakano [13]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$.

Definition 1.2. The double sequence $\lambda_2 = \{(\beta_r, \mu_s)\}$ is called double λ_2 sequence if there exist two non-decreasinig sequences of positive numbers tending to infinity such that $\beta_{r+1} \leq \beta_r + 1$, $\beta_1 = 1$ and $\mu_{s+1} \leq \mu_s + 1$, $\mu_1 = 1$. The generalized double de Vallee-Poussin mean is defined by

$$t_{rs} = t_{rs}(x_{mn}) = \frac{1}{\lambda_{rs}} \sum_{(m,n)\in I_{rs}} x_{mn},$$

where $\lambda_{rs} = \beta_r \cdot \mu_s$ and $I_{rs} = \{(mn) : r - \beta_r + 1 \leq m \leq r, s - \mu_s + 1 \leq n \leq s\}$.

A double number sequence $x = (x_{mn})$ is said to be (V_2, λ_2)-summable to a nunmber L if $P\text{-}\lim_{rs} t_{rs} = L$. If $\lambda_{rs} = rs$, then then (V_2, λ_2)-summability is reduced to $(C,1,1)$-summability.

§2. Main results

Let $A = \left(a_{i(k,\ell)}^{i(m,n)}\right)$ is an infinite four dimensional matrix of complex numbers and $p = \left(p_{i(mn)}\right)$ be a double analytic sequence of positive real numbers such that $0 < h = \inf_i p_{i(mn)} \leq$

$\sup_i p_{i(mn)} = H < \infty$ and f be a modulus. We define

$$\begin{aligned}
& V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right] \\
= & \left\{x = (x_{mn}) \in w^2 : lim_{r,s\to\infty}\lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} = 0\right\},
\end{aligned}$$

$$\begin{aligned}
& V_{2\Lambda^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right] \\
= & \left\{x = (x_{mn}) \in w^2 : sup_{r,s}\lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|A_i\left(\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} < \infty\right\},
\end{aligned}$$

where $A_i\left(\Delta^m x\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{i(k,\ell)}^{i(mn)} \Delta^m x_{mn}$.

Theorem 2.1. Let f be a modulus function. Then $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ is a linear space over the complex field $\mathbb{C}$.

Proof. Let $x,\ y \in V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ and $\alpha,\ \mu \in \mathbb{C}$. Then there exists integers D_α and D_μ such that $|\alpha|^{\frac{1}{m+n}} \leq D_\alpha$ and $|\mu|^{\frac{1}{m+n}} \leq D_\mu$. By using (1) and the properties of modulus f, we have

$$\begin{aligned}
& \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{i(k,\ell)}^{i(mn)} \left((m+n)!\Delta^m\left(\alpha x_{mn} + \mu x_{mn}\right)\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} \\
\leq\ & DD_\alpha^H \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha^{\frac{1}{m+n}} a_{i(k,\ell)}^{i(mn)} \left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} \\
& + DD_\mu^H \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha^{\frac{1}{m+n}} a_{i(k,\ell)}^{i(mn)} \left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} \\
\to\ & 0, \quad \text{as } r, s \to \infty.
\end{aligned}$$

This proves that $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ is linear. This comples the proof.

Theorem 2.2. Let f be a modulus function. Then the inclusions $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right] \subset V_{2\Lambda^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ hold.

Proof. Let $x \in V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ such that $x \to \left(V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]\right)$. By using (2), we have

$$\begin{aligned}
& \sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} \\
=\ & \sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} - 0 + 0\right|\right)\right]^{p_{i(mn)}} \\
\leq\ & D\sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} - 0\right|\right)\right]^{p_{i(mn)}} \\
& + D\sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(|0|\right)\right]^{p_{i(mn)}} \\
\leq\ & D\sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} - 0\right|\right)\right]^{p_{i(mn)}} \\
& + T\max\left\{f\left(|0|\right)^h, f\left(|0|\right)^H\right\} \\
<\ & \infty.
\end{aligned}$$

Hence $x \in V_{2\Lambda^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$. Therefore the inclusion $V_{\chi^2}^{\lambda}\left[A, \Delta^m, p, f\right] \subset V_{2\Lambda^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ holds. This completes the proof.

Theorem 2.3. Let $p = \left(p_{i(mn)}\right) \in \Lambda^2$. Then $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ is a paranormed space

$$g\left(x\right) = \sup_{rs}\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}},$$

where $M = \max\left(1, \sup_i p_i\right)$

Proof. Clearly $g\left(-x\right) = g\left(x\right)$. It is trivial that $\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} = 0$ for $x_{mn} = 0$. Hence we get $g\left(0\right) = 0$. Since $\frac{p_i}{M} \leq 1$ and $M \geq 1$, using Minkowski's inequality and definition of modulus f, for each $\left(r, s\right)$, we have

$$\begin{aligned}
&\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m\left(x_{mn}+y_{mn}\right)\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}\\
\leq\ &\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}\\
&+\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m y_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}.
\end{aligned}$$

Now it follows that g is subadditive. Let us take any complex number α. By definition of modulus f, we have

$$\begin{aligned}
g\left(\alpha x\right) &= \sup_{rs}\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m \alpha x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}\\
&\leq K^{\frac{H}{M}} g\left(x\right),
\end{aligned}$$

where $K = 1 + \left[\left|\alpha\right|^{\frac{1}{m+n}}\right]$ ($[|t|]$ denotes the integer part of t). Since f is modulus, we have $x \to 0$ implies $g\left(\alpha x\right) \to 0$. Similarly $x \to 0$ and $\alpha \to 0$ implies $g\left(\alpha x\right) \to 0$. Finally, we have x fixed and $\alpha \to 0$ implies $g\left(\alpha x\right) \to 0$. This comples the proof.

Theorem 2.4. Let f be a modulus function. Then $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p\right] \subset V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$.

Proof. Let $x \in V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p\right]$. We can choose $0 < \delta < 1$ such that $f\left(t\right) < \epsilon$ for every $t \in [0, \infty)$ with $0 \leq t \leq \delta$. Then, we can write

$$\begin{aligned}
&\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} - 0\right|\right)\right]^{p_{i(mn)}}\\
=\ &\lambda_{rs}^{-1}\sum_{\substack{mn\,\in\, Irs\\ \left|A_i\left((m+n)!\Delta^m xmn\right)^{\frac{1}{m+n}}\right|\leq\delta}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\\
&+\lambda_{rs}^{-1}\sum_{\substack{mn\,\in\, Irs\\ \left|A_i\left((m+n)!\Delta^m xmn\right)^{\frac{1}{m+n}}\right|>\delta}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\\
\leq\ &\max\left\{f\left(\epsilon\right)^h, f\left(\epsilon\right)^H\right\} + \max\left\{1, \left(2f\left(1\right)\delta^{-1}\right)^H\right\}\lambda_{rs}^{-1}
\end{aligned}$$

$$* \sum_{\substack{mn \in Irs \\ \left|A_i\left((m+n)!\Delta^m xmn\right)^{\frac{1}{m+n}}\right| > \delta}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}.$$

Therefore $x \in V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$. This completes the proof.

Theorem 2.5. Let $0 < p_{i(mn)} < q_{i(mn)}$ and $\left\{\frac{q_{i(mn)}}{p_{i(mn)}}\right\}$ be bounded. Then $V_{2\chi^2}^{\lambda}\left[A, \Delta^m, q, f\right] \subset V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$.

Proof. Let

$$x \in V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, q, f\right], \tag{2}$$

$$\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}} \to 0 \quad \text{as } r, s \to \infty. \tag{3}$$

Let $t_i = \lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}} \to 0$ as $r, s \to \infty$ and $\gamma_{i(mn)} = \frac{p_{i(mn)}}{q_{i(mn)}}$. Since $p_{i(mn)} \leq q_{i(mn)}$, we have $0 \leq \gamma_{i(mn)} \leq 1$. Take $0 < \gamma < \gamma_{i(mn)}$. Define $u_i = t_i (t_i \geq 1)$, $u_i = 0 (t_i < 1)$ and $v_i = 0 (t_i \geq 1)$, $v_i = t_i (t_i < 1)$. $t_i = u_i + v_i$, $t_i^{\gamma_{i(mn)}} = u_i^{\gamma_{i(mn)}} + v_i^{\gamma_{i(mn)}}$. Now it follows that

$$u_i^{\gamma_{i(mn)}} \leq u_i \leq t_i \text{ and } v_i^{\gamma_{i(mn)}} \leq v_i^{\gamma}. \tag{4}$$

i.e., $t_i^{\gamma_{i(mn)}} = u_i^{\gamma_{i(mn)}} + v_i^{\gamma_{i(mn)}}$, $t_i^{\gamma_{i(mn)}} \leq t_i + v_i^{\lambda}$ by (4),

$$\begin{aligned} &\left(\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}\right)^{\gamma_{i(mn)}} \\ \leq\ &\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}, \\ &\left(\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}\right)^{\frac{p_{i(mn)}}{q_{i(mn)}}} \\ \leq\ &\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}, \\ &\left(\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right) \\ \leq\ &\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}. \end{aligned}$$

But $\left(\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}\right) \to 0$ as $r, s \to \infty$. By (3), therefore $\left(\lambda_{rs}^{-1} \sum_{mn \in I_{rs}} \left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right) \to 0$ as $r, s \to \infty$. Hence

$$x \in V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]. \tag{5}$$

From (2) and (5), we get $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, q, f\right] \subset V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$. This completes the proof.

Theorem 2.6. (i) Let $0 < \inf p_i \leq p_i \leq 1$. Then $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right] \subset V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, f\right]$,

(ii) Let $1 \leq p_i \leq \sup p_i < \infty$. Then $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, f\right] \subset V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right]$,

(iii) Let $0 < p_i \leq q_i < \infty$ for each i. Then $V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, p, f\right] \subset V_{2\chi^2}{}^{\lambda_2}\left[A, \Delta^m, q, f\right]$.

Proof. The proof is a routine verification.

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Certain differential subordination involving a multiplier transformation

Sukhwinder Singh Billing

Department of Applied Sciences, Baba Banda Singh Bahadur Engineering College,
Fatehgarh Sahib, 140407, Punjab, India
E-mail: ssbilling@gmail.com

Abstract We, here, study a certain differential subordination involving a multiplier transformation which unifies some known differential operators. As a special case of our main result, we find some new results providing the best dominant for $f(z)/z^p,\ f(z)/z$ and $f'(z)/z^{p-1},\ f'(z)$.

Keywords Differential subordination, multiplier transformation, analytic function, univalent function, p-valent function.

§1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions that $f(0) = f'(0) - 1 = 0$. Thus, $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form $f(z) = z^p + \sum\limits_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N} = \{1, 2, 3, \cdots\}$, which are analytic and multivalent in the open unit disk $\mathbb{E}$. Note $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \geq 0, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et al.[1]. $I_1(n, 0)$ is the well-known Sălăgean [6] derivative operator D^n, defined for $f \in \mathcal{A}$ as under:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

For two analytic functions f and g in the unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for each dominant q of (1), is said to be the best dominant of (1).

Owa [5], studied the class $\mathcal{B}(\alpha, \beta)$ of functions $f \in \mathcal{A}$ satisfying the following inequality

$$\Re\left(\frac{f'(z)(f(z))^{\alpha-1}}{z^{\alpha-1}}\right) > \beta, \ z \in \mathbb{E}.$$

Owa [5], proved that if $f \in \mathcal{B}(\alpha, \beta)$, then $\Re\left(\frac{f(z)}{z}\right)^{\alpha} > \frac{1+2\alpha\beta}{1+2\alpha}, \ z \in \mathbb{E}.$

Liu [3], studied the class $\mathcal{B}(\lambda, \alpha, p, A, B)$ which is analytically defined as under:

$$\mathcal{B}(\lambda, \alpha, p, A, B) = \left\{ f \in \mathcal{A}_p : (1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \lambda \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \right\},$$

where $\alpha > 0, \ \lambda \geq 0, \ -1 \leq B \leq 1$ and $A \neq B$.

Liu [3], investigated the class $\mathcal{B}(\lambda, \alpha, p, A, B)$ to find the dominant F such that

$$\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec F(z),$$

whenever $f \in \mathcal{B}(\lambda, \alpha, p, A, B)$.

As a special case of our main result, we, here, obtain the function h such that $f \in \mathcal{A}_p$, satisfies

$$(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \lambda \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec h(z), \ z \in \mathbb{E},$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}, \ -1 \leq B < A \leq 1.$$

Lecko [2] also made some estimates on $\frac{f(z)}{z}$, $f \in \mathcal{A}$ in terms of certain differential subordinations.

In the present paper, we study a certain differential subordination involving the multiplier transformation $I_p(n, \lambda)$, defined above. The differential operator studied here, unifies the above mentioned differential operators. Our results generalize and improve some know results. We also obtain certain new results.

To prove our main result, we shall make use of the following lemma of Miller and Macanu [4].

Lemma 1.1. Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re\,\frac{zh'(z)}{Q(z)} > 0,\ z \in \mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

§2. Main results

In what follows, all the powers taken are the principle ones.

Theorem 2.1. Let α and β be non-zero complex numbers such that $\Re\,(\beta/\alpha) > 0$ and let $f \in \mathcal{A}_p,\ \left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta \left[1-\alpha+\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2}, \tag{2}$$

then

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta \prec \frac{1+Az}{1+Bz},\quad -1 \le B < A \le 1,\ z \in \mathbb{E}.$$

The dominant $\frac{1+Az}{1+Bz}$ is the best one.

Proof. Write $u(z) = \left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta$. A little calculation yields that

$$\left(\frac{I_p(n,\lambda)f(z)}{z^p}\right)^\beta \left[1-\alpha+\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] = u(z) + \frac{\alpha}{\beta(p+\lambda)}zu'(z). \tag{3}$$

Define the functions θ and ϕ as under:

$$\theta(w) = w \quad\text{and}\quad \phi(w) = \frac{\alpha}{\beta(p+\lambda)}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C}$ and $\phi(w) \neq 0,\ w \in \mathbb{D}$. Setting $q(z) = \frac{1+Az}{1+Bz},\ -1 \le B < A \le 1,\ z \in \mathbb{E}$ and defining the functions Q and h as follows:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha}{\beta(p+\lambda)}zq'(z) = \frac{\alpha}{\beta(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha}{\beta(p+\lambda)}zq'(z) = \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta(p+\lambda)}\frac{(A-B)z}{(1+Bz)^2}. \tag{4}$$

A little calculation yields

$$\Re\left(\frac{zQ'(z)}{Q(z)}\right) = \Re\left(1+\frac{zq''(z)}{q'(z)}\right) = \Re\left(\frac{1-Bz}{1+Bz}\right) > 0,\ z \in \mathbb{E},$$

i.e., Q is starlike in $\mathbb{E}$ and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(1 + \frac{zq''(z)}{q'(z)} + (p+\lambda)\frac{\beta}{\alpha}\right) = \Re\left(\frac{1-Bz}{1+Bz}\right) + (p+\lambda)\Re\left(\frac{\beta}{\alpha}\right) > 0,\ z \in \mathbb{E}.$$

Thus conditions (ii) and (iii) of Lemma 1.1, are satisfied. In view of (2), (3) and (4), we have

$$\theta[u(z)] + zu'(z)\phi[u(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof follows from Lemma 1.1.

For $p = 1$ and $\lambda = 0$ in above theorem, we get the following result involving Sălăgean operator.

Theorem 2.2. If $\alpha,\ \beta$ are non-zero complex numbers such that $\Re(\beta/\alpha) > 0$. If $f \in \mathcal{A},\ \left(\frac{D^n f(z)}{z}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, satisfies

$$\left(\frac{D^n f(z)}{z}\right)^{\beta}\left[1 - \alpha + \alpha\frac{D^{n+1} f(z)}{D^n f(z)}\right] \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2},\ \ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{D^n f(z)}{z}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

The dominant $\frac{1+Az}{1+Bz}$ is the best one.

§3. Dominant for $f(z)/z^p,\ f(z)/z$

In this section, we obtain the best dominant for $f(z)/z^p$ and $f(z)/z$, by considering particular cases of main result. Select $\lambda = n = 0$ in Theorem 2.1, we obtain:

Corollary 3.1. Suppose $\alpha,\ \beta$ are non-zero complex numbers such that $\Re(\beta/\alpha) > 0$ and if $f \in \mathcal{A}_p,\ \left(\frac{f(z)}{z^p}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, satisfies

$$(1-\alpha)\left(\frac{f(z)}{z^p}\right)^{\beta} + \alpha\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p\beta}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ \ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $\beta = 1$ in above theorem, we obtain:

Corollary 3.2. Suppose α is a non-zero complex number such that $\Re(1/\alpha) > 0$. If $f \in \mathcal{A}_p,\ \frac{f(z)}{z^p} \neq 0,\ z \in \mathbb{E}$, satisfies

$$(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\frac{f(z)}{z^p} \prec \frac{1+Az}{1+Bz},\ \ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

On writing $\alpha = 1$ in Corollary 3.1, we get:

Corollary 3.3. Let β be a complex number with $\Re(\beta) > 0$ and let $f \in \mathcal{A}_p$, $\left(\frac{f(z)}{z^p}\right)^\beta \neq 0,\ z \in \mathbb{E}$, satisfy

$$\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^\beta \prec \frac{1+Az}{1+Bz} + \frac{1}{p\beta}\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{f(z)}{z^p}\right)^\beta \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Selecting $\alpha = \beta = 1/2$ in Corollary 3.1, we get:

Corollary 3.4. If $f \in \mathcal{A}_p$, $\sqrt{\frac{f(z)}{z^p}} \neq 0,\ z \in \mathbb{E}$, satisfies

$$\sqrt{\frac{f(z)}{z^p}}\left(1 + \frac{zf'(z)}{pf(z)}\right) \prec \frac{2(1+Az)}{1+Bz} + \frac{2}{p}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\sqrt{\frac{f(z)}{z^p}} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $p = 1$ in Corollary 3.2, we have the following result.

Corollary 3.5. If α is a non-zero complex number such that $\Re(1/\alpha) > 0$ and if $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0,\ z \in \mathbb{E}$, satisfies

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \frac{1+Az}{1+Bz} + \alpha\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Writing $\alpha = 1$ in above corollary, we obtain:

Corollary 3.6. If $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0,\ z \in \mathbb{E}$, satisfies

$$f'(z) \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\frac{f(z)}{z} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Setting $p = 1$ in Corollary 3.3, we have the following result:

Corollary 3.7. If β is a complex number with $\Re(\beta) > 0$ and if $f \in \mathcal{A}$, $\left(\frac{f(z)}{z}\right)^\beta \neq 0,\ z \in \mathbb{E}$, satisfies

$$\frac{f'(z)(f(z))^{\beta-1}}{z^{\beta-1}} \prec \frac{1+Az}{1+Bz} + \frac{1}{\beta}\frac{(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{f(z)}{z}\right)^\beta \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{E}.$$

Setting $p = 1$ in Corollary 3.1, we obtain the following result that corresponds to the main result of Shanmugam et al. [7].

Corollary 3.8. If $\alpha,\ \beta$ are non-zero complex numbers such that $\Re(\beta/\alpha) > 0$. If $f \in \mathcal{A},\ \left(\frac{f(z)}{z}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, satisfies

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

§4. Dominant for $f'(z)/z^{p-1},\ f'(z)$

In this section, we find the best dominant for $f'(z)/z^{p-1}$ and $f'(z)$ as special cases of main result. Select $\lambda = 0$ and $n = 1$ in Theorem 2.1, we obtain:

Corollary 4.1. Let $\alpha,\ \beta$ be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$ and let $f \in \mathcal{A}_p,\ \left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, satisfy

$$(1-\alpha)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} + \frac{\alpha}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p\beta}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\left(\frac{f'(z)}{pz^{p-1}}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Taking $\beta = 1$ in above theorem, we obtain:

Corollary 4.2. Suppose α is a non-zero complex number such that $\Re(1/\alpha) > 0$. If $f \in \mathcal{A}_p,\ \frac{f'(z)}{pz^{p-1}} \neq 0,\ z \in \mathbb{E}$, satisfies

$$(1-\alpha)\frac{f'(z)}{pz^{p-1}} + \frac{\alpha}{p^2}\frac{f'(z)}{z^{p-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p}\frac{(A-B)z}{(1+Bz)^2},\ z \in \mathbb{E},$$

then

$$\frac{f'(z)}{z^{p-1}} \prec \frac{p(1+Az)}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

On writing $\alpha = 1$ in above corollary, we get:

Corollary 4.3. If $f \in \mathcal{A}_p,\ \frac{f'(z)}{pz^{p-1}} \neq 0,\ z \in \mathbb{E}$, satisfies

$$\frac{f'(z)}{z^{p-1}} + \frac{f''(z)}{z^{p-2}} \prec \frac{p^2(1+Az)}{1+Bz} + \frac{p(A-B)z}{(1+Bz)^2},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E},$$

then

$$\frac{f'(z)}{z^{p-1}} \prec \frac{p(1+Az)}{1+Bz},\ z \in \mathbb{E}.$$

Taking $p = 1$ in Corollary 4.2, we have the following result:

Corollary 4.4. If α is a non-zero complex number such that $\Re(1/\alpha) > 0$ and if $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies

$$f'(z) + \alpha z f''(z) \prec \frac{1+Az}{1+Bz} + \alpha\frac{(A-B)z}{(1+Bz)^2}, \ -1 \leq B < A \leq 1, \ z \in \mathbb{E},$$

then

$$f'(z) \prec \frac{1+Az}{1+Bz}, \ z \in \mathbb{E}.$$

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Modified multi-step Noor method for a finite family of strongly pseudo-contractive maps

Adesanmi Alao Mogbademu

Department of Mathematics, University of Lagos, Nigeria

Abstract A convergence results for a family of strongly pseudocontractive maps in Banach spaces is proved when at least one of the mappings is strongly pseudocontractive under some mild conditions, using a new iteration formular. Our results represent an improvement of previously known results.

Keywords Noor iteration, strongly accretive mappings, strongly pseudocontractive mappin -gs, strongly Φ-pseudocontractive operators, Banach spaces.
2000 AMS Mathematics Classification: 47H10, 46A03.

§1. Introduction

Let X be an arbitrary real Banach Space and let $J: X \to 2^{X^*}$ be the normalized duality mapping defined by

$$J(x) = \{f \in X^* : < x, f >= \|x\|^2 = \|f\|^2\}, \ \forall x \in X, \tag{1}$$

where X^* denotes the dual space of X and $< .,. >$ denotes the generalized duality pairing between X and X^*. The single-valued normalized duality mapping is denoted by j.

Let K be a nonempty subset of X. A map $T: K \to K$ is strongly pseudocontractive if there exists $k \in (0, 1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2, \ \forall x, y \in K. \tag{2}$$

A map $T: K \to K$ is strongly accretive if there exists $k \in (0, 1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geq k\|x-y\|^2, \ \forall x, y \in K. \tag{3}$$

In (2), take $k = 1$ to obtain a pseudocontractive map. In (3), take $k = 0$ to obtain an accretive map.

Recently, Zhang [4] studied convergence of Ishikawa iterative sequence for strongly pseudocontractive operators in arbitrary Banach spaces under the condition of removing the restriction of any boundedness. In fact, he proved the following theorem:

Theorem 1.1. Let X be a real Banach space, K a non-empty, convex subset of X and let T be a continuous and strongly pseudocontractive self mappings with a pseudocontractive

parameter $k \in (0,\ 1)$. For arbitrary $x_0 \in K$, let Ishikawa iteration sequence $\{x_n\}_0^\infty$ be defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{4}$$

where $\alpha_n,\ \beta_n \in [0,\ 1]$, and constants $a,\ \tau \in (0,\ 1-k)$ are such that

$$0 < a \leq \alpha_n < 1 - k - \tau,\ n \geq 0, \tag{5}$$

If $\|Ty_n - Tx_{n+1}\| \to 0$ as $n \to \infty$, then the sequence $\{x_n\}_0^\infty$ converges strongly to a unique fixed point of $T \in K$; Moreover,

$$\|x_n - \rho\| \leq ((1-a\tau)^n \|x_0 - \rho\|^2 + (\frac{(1-(1-a\tau)^n)M}{a\tau})^{\frac{1}{2}}, \quad n \geq 1,$$

where $M = sup \frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2$.

This result itself is a generalization of many of the previous results (see [4] and the references therein).

Let $p \geq 2$ be fixed. Let $T_i:\ K \to K$, $1 \leq i \leq p$, be a family of maps. We introduce the following modified multi-step Noor iteration:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n^1, \\ y_n^i &= (1-\alpha_n^i)x_n + \alpha_n^i T_{1+i} y_n^{1+i}, \quad i = 1, \cdots, p-2, \\ y_n^{p-1} &= (1-\alpha_n^{p-1})x_n + \alpha_n^{p-1} T_p x_n, \end{aligned} \tag{6}$$

where the sequences $\{\alpha_n\},\ \{\alpha_n^i\}$ $(i = 1, \cdots, p-2)$, in $[0,\ 1)$ satisfies certain conditions. It is clear that the iteration defined by (6) is a generalization of the Ishikawa iteration (4).

Let $F(T_1, \cdots, T_p)$ denote the common fixed points set with respect to K for the family $T_1, \cdots, T_p$. In this paper, following the method of proof of Zhang [4], we prove a convergence results for iteration (6), for strongly pseudocontractive maps when the iterative parameter α_n satisfies (5). These results extend and equally improve the recently obtained results from [4]. We give numerical example to demostrate that the modified multi-step Noor iteration (6) converges faster than the Ishikawa iteration [1].

Lemma 1.1.[2] Let X be real Banach Space and $J:\ X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x,\ y \in X$,

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y,\ j(x+y) >,\ \forall j(x+y) \in J(x+y).$$

Lemma 1.2.[3] Let $\{\alpha_n\}$ be a non- negative sequence which satisfies the following inequality

$$\rho_{n+1} \leq (1-\lambda)\rho_n + \delta_n,$$

where $\lambda_n \in (0,\ 1)$, $\forall n \in N$, $\sum_{n=1}^\infty \lambda_n = \infty$ and $\delta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \alpha_n = 0$.

§2. Main results

Theorem 2.1. Let $p \geq 2$ be fixed, X be a real Banach space and K a non-empty, convex subset of X. Let T_1 be a strongly pseudocontractive map and $T_2, \cdots, T_p : K \to K$, with pseudocontractive parameter $k \in (0,\ 1)$, such that $F(T_1, \cdots, T_p) \neq \phi$. If $a,\ \tau \in (0,\ 1-k)$, $\alpha_n \in [0,\ 1)$ satisfies (5), $x_0 \in K$, and the following condition is satisfied:

$$\lim_{n\to\infty} \|Ty_n - Tx_{n+1}\| = 0, \tag{7}$$

then the iteration (6) converges strongly to a unique common fixed point of $T_1, \cdots, T_p$, which is the unique fixed point of T_1.

Moreover,

$$\|x_n - \rho\| \leq ((1-2a\tau)^n \|x_0 - \rho\|^2 + (\frac{(1-(1-2a\tau)^n)M}{2a\tau})^{\frac{1}{2}}, \quad n \geq 0,$$

where $M = sup\{\frac{1}{(k+\tau)^2}\|T_1 y_n - T_1 x_{n+1}\|^2\}$.

Proof. Since T_1 is strongly pseudocontractive, then there exists a constant k so that

$$\langle T_1 x - T_1 y,\ j(x-y)\rangle \leq k\|x-y\|^2,$$

Let ρ be such that $T_1\rho = \rho$. From Lemma 1.2, we have

$$\begin{aligned}
\|x_{n+1} - \rho\|^2 &= \langle x_{n+1} - \rho, j(x_{n+1} - \rho)\rangle \\
&= \langle (1-\alpha_n)x_n + \alpha_n T_1 y_n - (1-\alpha_n)\rho - \alpha_n, j(x_{n+1} - \rho)\rangle \\
&= \langle (1-\alpha_n)(x_n - \rho) + \alpha_n(T_1 y_n - \rho), j(x_{n+1} - \rho)\rangle \\
&= \langle (1-\alpha_n)(x_n - \rho), j(x_{n+1} - \rho)\rangle \\
&\quad + \langle \alpha_n(T_1 y_n - \rho), j(x_{n+1})\rangle \\
&= (1-\alpha_n)\langle x_n - \rho, j(x_{n+1} - \rho)\rangle \\
&\quad + \alpha_n\langle T_1 y_n - T_1 x_{n+1}, j(x_{n+1} - \rho)\rangle \\
&\quad + \alpha_n\langle T_1 x_{n+1} - \rho, j(x_{n+1} - \rho)\rangle.
\end{aligned} \tag{8}$$

By strongly pseudocontractivity of T_1, we get

$$\alpha_n\langle T_1 x_{n+1} - \rho,\ j(x_{n+1})\rangle \leq \alpha_n k\|x_{n+1} - \rho\|^2,$$

for each $j(x_{n+1} - \rho) \in J(x_{n+1} - \rho)$, and a constant $k \in (0,\ 1)$.

From inequality (8) and inequality $ab \leq \frac{a^2+b^2}{2}$, we obtain that

$$(1-\alpha_n)\|x_n - \rho\|\|x_{n+1} - \rho\| \leq \frac{1}{2}(1-\alpha_n)\|x_n - \rho\|^2 + \|x_{n+1} - \rho\|^2 \tag{9}$$

and

$$\alpha_n\|T_1 y_n - T_1 x_{n+1}\|\|x_{n+1} - \rho\| \leq \frac{1}{2}(\|T_1 y_n - T_1 x_{n+1}\|^2 + \alpha_n^2\|x_{n+1} - \rho\|^2). \tag{10}$$

Substituting (9) and (10) into (8), we infer that

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &\leq \frac{1}{2}((1-\alpha_n)^2\|x_n-\alpha_n\|^2+\|x_{n+1}-\rho\|^2)\\ &+\frac{1}{2}(\|T_1y_n-T_1x_{n+1}\|^2+\alpha_n^2\|x_{n+1}-\rho\|^2)\\ &+\alpha_n k\|x_{n+1}-\rho\|^2. \end{aligned}\tag{11}$$

Multiplying inequality (11) by 2 throughout,we have

$$\begin{aligned}2\|x_{n+1}-\rho\|^2 &\leq (1-\alpha_n)^2\|x_n-\rho\|^2+\|x_{n+1}-\rho\|^2\\ &+\|T_1y_n-T_1x_{n+1}\|^2+\alpha^2\|x_{n+1}-\rho\|^2\\ &+2\alpha_n k\|x_{n+1}-\rho\|^2. \end{aligned}\tag{12}$$

By collecting like terms $\|x_{n+1}-\rho\|^2$ and simplifying, we have

$$(1-2\alpha_n k-\alpha_n^2)\|x_{n+1}-\rho\|^2\leq(1-\alpha_n)^2\|x_n-\rho\|^2+\|T_1y_n-T_1x_{n+1}\|^2,$$

using (5), we obtain

$$\begin{aligned}(1-\alpha_n)^2 &\leq 1-2\alpha_n+\alpha_n(1-k-\tau)=1-\alpha_n-k\tau\\ &< 1-2\alpha_n k-\alpha_n\tau<1-2\alpha_n k-\alpha_n^2, \end{aligned}\tag{13}$$

thus

$$\begin{aligned}\|x_{n+1}-\rho\|^2 &\leq \frac{(1-\alpha_n)^2}{1-2\alpha_n k-\alpha_n^2}\|x_n-\rho\|^2\\ &+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}, \end{aligned}\tag{14}$$

for all $n\geq 0$.

Since $k,\ \alpha_n\in(0,\ 1)$ and constants $a,\ \tau\in(0,\ 1-k)$ we have

$$1-\alpha_n>k+\tau. \tag{15}$$

From (13), one can have

$$\frac{1}{1-2\alpha_n k-\alpha_n^2}<\frac{1}{(1-\alpha_n)^2}<\frac{1}{(k+\tau)^2}. \tag{16}$$

From the above and (14) we have

$$
\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq \frac{(1-\alpha_n)^2}{1-2\alpha_n k-\alpha_n^2}\|x_n-\rho\|^2+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&= \frac{(1-\alpha_n)^2(1-\alpha_n(2k+\alpha_n)+\alpha_n(2k+\alpha_n))}{1-2\alpha_n k-\alpha_n^2}\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&= (1-\alpha_n)^2(1+\frac{\alpha_n(2k+\alpha_n)}{1-2\alpha_n k-\alpha_n^2})\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&= ((1-\alpha_n)^2+\frac{\alpha_n(2k+\alpha_n)(1-\alpha_n)^2}{1-2\alpha_n k-\alpha_n^2})\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-\alpha_n)^2+\alpha_n(2k+\alpha_n)\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&= (1-2\alpha_n(1-k-\alpha_n))\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{1-2\alpha_n k-\alpha_n^2}\\
&\leq (1-2\alpha_n(1-k-(1-k-\tau)))\|x_n-\rho\|^2\\
&\quad+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{(k+\tau)^2}\\
&= (1-2\alpha_n\tau)\|x_n-\rho\|^2+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{(k+\tau)^2}\\
&\leq (1-2a\tau)\|x_n-\rho\|^2+\frac{\|T_1y_n-T_1x_{n+1}\|^2}{(k+\tau)^2}, \qquad (17)
\end{aligned}
$$

for all $n \geq 0$. Set $\lambda = 2a\tau$, $\rho_n = \|x_n-\rho\|^2$, $\delta_n = \frac{\|T_1y_n-T_1x_{n+1}\|^2}{(k+\tau)^2}$.

Lemma 1.2 ensures that $x_n \to \rho$ as $n \to \infty$, that is, $\{x_n\}$ converges strongly to the unique fixed point ρ of the T_1.

Observe from inequality (17) that

$$
\begin{aligned}
\|x_1-p\|^2 &\leq (1-2a\tau)\|x_0-p\|^2+M\\
\|x_2-p\|^2 &\leq (1-2a\tau)\|x_1-p\|^2+M\\
&\leq (1-2a\tau)[(1-2a\tau)\|x_0-p\|^2+M]+M\\
&= (1-2a\tau)^2\|x_0-p\|^2+M+M(1-2a\tau)\\
&\quad .\\
&\quad .\\
&\quad .\\
\|x_n-p\|^2 &\leq (1-2a\tau)^n\|x_0-\rho\|^2+\tfrac{(1-(1-2a\tau)^n)}{2a\tau}M,
\end{aligned}
$$

for all $n \geq 0$, which implies that

$$\|x_n - \rho\| \leq ((1-2a\tau)\|x_0 - \rho\|^2 + \frac{(1-(1-2a\tau)^n)}{2a\tau}M)^{\frac{1}{2}},$$

where $M = sup\{\frac{1}{(k+\tau)^2}\|T_1 y_n - T_1 x_{n+1}\|^2\}$.
This completes the proof.

Remarks 2.1. Theorem 2.1 improves and extends Theorem 1 of Zhang [4] in the following sense:

(i) The continuity of the map is not necessary.

(ii) We replaced Ishikawa iterative process by a more general modified multi-step Noor iterative process.

(iii) We obtained a better convergence estimate.

We consider iteration (6), with $T_i x = f_i + (I - S_i)x,\ 1 \leq i \leq p$ and $p \geq 2$, I the identity operator, $\{\alpha_n\},\ \{\alpha_n^i\}$ $(i = 1, \cdots, p-2)$, in [0, 1) satisfying (5):

$$\begin{aligned}
&x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f_1 + (I - S_1)y_n^1),\\
&y_n^i = (1-\alpha_n^i)x_n + \alpha_n^i(f_{i+1} + (I - S_{i+1})y_n^{1+i}), \quad i = 1, \cdots, p-2,\\
&y_n^{p-1} = (1-\alpha_n^{p-1})x_n + \alpha_n^{p-1}(f_{p-1} + (I - S_p)x_n). \qquad (18)
\end{aligned}$$

Theorem 2.1 lead to the following result.

Theorem 2.2. Let $p \geq 2$ be fixed, X be a real Banach space , $T_1 :\ X \to X$ be a strongly pseudocontractive map and $S_2, \cdots, S_p :\ X \to X$, such that the equation $S_i x = f_i,\ 1 \leq i \leq p$, have a common solution. If $a,\ \tau \in (0,\ 1-k)$, $\alpha_n \in [0,\ 1)$ satisfies (5), and condition (7) is satisfied, then the iteration (18) converges strongly to a common solution of $S_i x = f_i,\ 1 \leq i \leq p$. Moreover,

$$\|x_n - \rho\| \leq ((1-2a\tau)^n\|x_0 - \rho\|^2 + (\frac{(1-(1-2a\tau)^n)M}{2a\tau})^{\frac{1}{2}}, \quad n \geq 0,$$

where $M = sup\{\frac{1}{(k+\tau)^2}\|T_1 y_n - T_1 x_{n+1}\|^2\}$.

§3. Numerical examples

Let K=[0, ∞) and X=($-\infty$, ∞) with the usual norm.

The map $T :\ K \to K$ is given by

$$Tx = \frac{x}{2(1+x)},\ \forall x \in K. \qquad (19)$$

Then the following can easily be verified:

(i) T is strongly pseudocontractive map.

(ii) $F(T) = \{0\}$.

We give an example where the modified multi-step Noor iteration (6) converges faster than the Ishikawa iteration (4) with $\{\alpha_n\}$ in both iterations, satisfying condition (5).

Case *. Consider $T_1 = T$, $T_2 = 2x$, $p = 2$ and the initial point x_0=0.5. Suppose $\{\alpha_n\}$ satisfies (5) such that $\alpha_n = 0.7$, $\forall n \in N$.

Using a Python program, we observe that our assumption (case *) converges faster and better towards the fixed point $\rho = 0$ as shown in below Table 1.

Table 1: **Numerical Results of Iteration**

Iteration	**modified Noor**	**Ishikawa**
Step 10	0.00402	0.11729
Step 15	0.00046	0.08388
Step 18	0.00012	0.07157
Step 19	0.00008	0.06822
Step 20	0.00005	0.06517
Step 25	0.0000	0.05324
Step 1000	-	0.00143

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On partial sums of generalized dedekind sums[1]

Hangzhou Ma

Department of Mathematics and Statistics, Xi'an Jiaotong University,
Xi'an, 710049, P. R. China
E-mail: hzmaths@hotmail.com

Abstract The main purpose of this paper is to use the mean value theorems of Dirichlet L-function to study the distribution properties of the generalized Dedekind sums, and gives two interesting asymptotic formulas.

Keywords Generalized Dedekind sums, partial sums, mean value.
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§1. Introduction

Let k be a positive integer, for arbitrary integers h and m, n, the generalized Dedekind sums $S(h,m,n,k)$ is defined by

$$S(h,m,n,k)=\sum_{a=1}^{k}\bar{B}_m\left(\frac{a}{k}\right)\bar{B}_n\left(\frac{ah}{k}\right),$$

where

$$\bar{B}_n(x)=\begin{cases}B_n(x-[x]), & \text{if } x \text{ is not an integer},\\ 0, & \text{if } x \text{ is an integer},\end{cases}$$

with $B_n(x)$ the Bernoulli polynomial. Some arithmetic theorems of generalized Dedekind sums have been studied in [1-2]. For simple examples:

For any positive number q, we have

$$S(qh,m,n,qk)=\frac{1}{q^{m-1}}S(h,m,n,k).$$

For a prime p, n an odd positive number, we have

$$\sum_{i=1}^{p-1}S(h+ik,m,n,pk)=\left(\frac{1}{p^{m+n-2}}+p\right)S(h,m,n,k)-\frac{1}{p^{n-1}}S(ph,m,n,k).$$

Particularly, when $p=2$, we have

$$S(h+k,m,n,2k)=\left(\frac{1}{2^{m+n-2}}+2\right)S(h,m,n,k)-\frac{1}{2^{n-1}}S(2h,m,n,k)-S(h,m,n,2k).$$

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Partial sums method plays an important role in modern number theory. The common types of sums$\sum_{n<x} f(n)$ are those in which f is a "smooth" function that is defined for real arguments x. In this paper, we use it in the process of estimating sums of Generalized Dedekind sums, namely

$$\sum_{h\leqslant N}{}' S(h,m,n,k).$$

The basic idea for handling such sums is to approximate the sum by a corresponding integral and investigate the error made in the process. The following important result, known as Euler's summation formula, gives an exact formula for the difference between such a sum and the corresponding integral.

Euler's summation formula. Let $0 < y \leq x$ and suppose $f(t)$ is a function defined on the interval $[y, x]$ and has a continuous derivative there. Then

$$\sum_{y<N\leq x}{}' f(n) = \int_y^x f(t)dt + \int_x^y tf^{'}(t)dt + \{y\}f(y) - \{x\}f(x),$$

where $\{t\}$ denotes the fractional part of t, i.e., $\{t\} = t - [t]$.

In 2006, Yiwei Hou studied the Dedekind Sums for $m = n$, namely $S(h, m, n, k) = S(h, n, k)$. They gave the following fomulas:

Let k be an integer with $k \geqslant 3$. Then for any positive real number N,

(i) If $n > 1$ is an odd number, then

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,n,k) &= \frac{(n!)^2}{2^{2n-1}\pi^{2n}}\zeta(2n)\zeta(n)k\prod_{p|k}(1-p^{-n})\\ &\quad +O\left(N^{-n}k^{2+\varepsilon} + N^{2n}k^{-2n+1+\varepsilon} + N^n k^{-n+1+\varepsilon}\right).\end{aligned}$$

(ii) If n is a positive even number, then

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,n,k) &= \frac{(n!)^2}{2^{2n-1}\pi^{2n}}\zeta(2n)\zeta(n)k\prod_{p|k}(1-p^{-n})\\ &\quad +O\left(N + N^{-n}k^{2+\varepsilon} + N^{2n}k^{-2n+1+\varepsilon} + N^n k^{-n+1+\varepsilon}\right).\end{aligned}$$

Together with the method in Yiwei Hou [10], the estimates of the character sums and the mean value theorems of Dirichlet L-function, we give two asymptotic formulas, namely

Theorem 1.1. Let k be an integer with $k \geqslant 3$. Then for any positive real number N, we have

(i) If $m > 1,\ n > 1$ are odd numbers, then

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,m,n,k) &= \frac{2m!n!}{(2i\pi)^{m+n+2}}\zeta(m+n)\zeta(m)k\prod_{p|k}(1-p^{-m})\\ &\quad +O\left(N^{-m}k^{2+\varepsilon} + N^{m+n}k^{-m-n+1+\varepsilon} + N^n k^{-n+1+\varepsilon}\right).\end{aligned}$$

(ii) If $m,\ n$ are positive even numbers, then

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,m,n,k) &= \frac{2m!n!}{(2i\pi)^{m+n}}\zeta(m+n)\zeta(m)k\prod_{p|k}(1-p^{-m})\\ &\quad +O\left(N + N^{-m}k^{2+\varepsilon} + N^{m+n}k^{-m-n+1+\varepsilon} + N^n k^{-n+1+\varepsilon}\right).\end{aligned}$$

Where $\sum_{h \leqslant N}{}'$ denotes the summation over all $h \leqslant N$ such that $(h, k) = 1$, $i^2 = -1$, ε is a sufficiently small positive real number which can be different at each occurrence. These results are obviously nontrivial for $k^{\varepsilon} < N < k^{1-\varepsilon}$.

§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 2.1. Let $k \geqslant 3$ be an integer. Then for any integer h with $(h, k) = 1$, we have the identities

(i) For $m,\ n$ positive odd numbers,

$$S(h, m, n, k) = \frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n}} \sum_{d|k} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \bar{\chi}(-h) L(m, \chi) L(n, \bar{\chi}).$$

(ii) For $m,\ n$ positive even numbers,

$$S(h, m, n, k) = \frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n}} \sum_{d|k} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=1}} \bar{\chi}(-h) L(m, \chi) L(n, \bar{\chi}) - \frac{4m!n!}{(2i\pi)^{m+n}} \zeta(m)\zeta(n).$$

Where χ denotes a Dirichlet character modulo d, $L(n, \chi)$ denotes the Dirichlet L-function corresponding to χ, $\phi(d)$ and $\zeta(s)$ are the Euler function and Riemann zeta-function, respectively.

Proof. See reference [9].

Lemma 2.2. Suppose that k, a and λ are positive integers, $q \geqslant 2$, $q|k$, then for any real number $q^{\varepsilon} < N < q^{1-\varepsilon}$ and any integers $t \geqslant s \geqslant 2$, we have the asymptotic formula

$$\begin{aligned} \sum_{\substack{a \leqslant N \\ (a,k)=1}} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^{\lambda}}} \bar{\chi}(a) L(s, \chi) L(t, \bar{\chi}) &= \frac{\phi(q)}{2} \zeta(s+t)\zeta(s) \prod_{p|q} (1 - p^{-s-t}) \prod_{p|k} (1 - p^{-s}) \\ &\quad + O\left(\phi(q) N^{-s+1} + \frac{\phi(q)}{q^t} q^{\varepsilon} N^t \right) \end{aligned}$$

Proof. (1) If $\lambda \equiv 1(\bmod 2)$, χ is an odd character mod q, Abel's identity implies that

$$L(s, \chi) = \sum_{n \leqslant q} \frac{\chi(n)}{n^s} + s \int_q^{+\infty} \frac{A(\chi, y)}{y^{s+1}} dy,$$

$$L(t, \bar{\chi}) = \sum_{m \leqslant \frac{q}{a}} \frac{\chi(m)}{m^t} + t \int_{\frac{q}{a}}^{+\infty} \frac{B(\bar{\chi}, z)}{z^{t+1}} dz,$$

where $A(\chi, y) = \sum_{q<n\leqslant y} \chi(n)$, $B(\bar{\chi}, z) = \sum_{\frac{q}{a}<m\leqslant z} \bar{\chi}(m)$. Thus

$$\begin{aligned}
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)L(s,\chi)L(t,\bar{\chi}) &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s} + s\int_q^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\\
&\quad \times\left(\sum_{m\leqslant \frac{q}{a}}\frac{\overline{\chi}(m)}{m^t} + t\int_{\frac{q}{a}}^{+\infty}\frac{B(\overline{\chi},z)}{z^{t+1}}dz\right)\\
&= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\sum_{m\leqslant \frac{q}{a}}\frac{\overline{\chi}(m)}{m^t}\right)\\
&\quad +t\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\int_{\frac{q}{a}}^{+\infty}\frac{B(\overline{\chi},z)}{z^{t+1}}dz\right)\\
&\quad +s\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{m\leqslant \frac{q}{a}}\frac{\overline{\chi}(m)}{m^t}\right)\left(\int_q^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\\
&\quad +st\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\int_q^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\left(\int_{\frac{q}{a}}^{+\infty}\frac{B(\overline{\chi},z)}{z^{t+1}}dz\right)\\
&= M_1+M_2+M_3+M_4
\end{aligned}$$

say. We then have

$$\sum_{\substack{a\leqslant N\\ (a,k)=1}}\ \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(\bar{a})L(s,\chi)L(t,\bar{\chi}) \equiv \sum_{\substack{a\leqslant N\\ (a,k)=1}} (M_1+M_2+M_3+M_4). \tag{1}$$

We need to estimate M_1, M_2, M_3 and M_4 respectively.

(i) For $(q, mn) = 1$,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \chi(n)\overline{\chi}(m) = \begin{cases} \frac{1}{2}\phi(q), & \text{if } \ n\equiv m \bmod q,\\ -\frac{1}{2}\phi(q), & \text{if } \ n\equiv -m \bmod q,\\ 0, & \text{otherwise.}\end{cases}$$

We can deduce that when $a \geqslant 2$,

$$\begin{aligned}
M_1 &= \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\sum_{m\leqslant \frac{q}{a}}\frac{\overline{\chi}(m)}{m^t}\right)\\
&= \frac{1}{2}\phi(q)\sideset{}{'}\sum_{\substack{n\leqslant q\\ n\equiv ma \bmod q}}\ \sideset{}{'}\sum_{m\leqslant \frac{q}{a}}\frac{1}{n^s m^t} - \frac{1}{2}\phi(q)\sideset{}{'}\sum_{\substack{n\leqslant q\\ n\equiv -ma \bmod q}}\ \sideset{}{'}\sum_{m\leqslant \frac{q}{a}}\frac{1}{n^s m^t}\\
&= \frac{1}{2}\phi(q)\sideset{}{'}\sum_{m\leqslant \frac{q}{a}}\frac{1}{m^t(ma)^s} - \frac{1}{2}\phi(q)\sideset{}{'}\sum_{m\leqslant \frac{q}{a}}\frac{1}{m^t(q-ma)^s}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\phi(q)}{2a^s}\sum_{m=1}^{\infty}{}' \frac{1}{m^{t+s}} + O\left(\frac{\phi(q)}{a^s}\sum_{m>\frac{q}{a}}\frac{1}{m^{s+t}}\right) + O\left(\phi(q)\sum_{m\leqslant\frac{q}{2a}}\frac{1}{m^t q^s}\right)\\
&\quad + O\left(\phi(q)\sum_{\frac{q}{2a}<m\leqslant\frac{q}{a}-1}\frac{a^t}{q^t(q-ma)^s}\right) + O\left(\frac{\phi(q)a^t}{q^t(q-a[\frac{q}{a}])^s}\right)\\
&= \frac{\phi(q)}{2a^s}\sum_{m=1}^{\infty}{}' \frac{1}{m^{t+s}} + O\left(\frac{\phi(q)}{q^s}\right) + O\left(\frac{\phi(q)}{q^s}\right)\\
&\quad + O\left(\frac{\phi(q)a^t}{q^t}(a^{-s}+(\frac{q}{2})^{-s})\right) + O\left(\frac{\phi(q)a^t}{q^t(q-a[\frac{q}{a}])^s}\right)\\
&= \frac{\phi(q)}{2a^s}\sum_{m=1}^{\infty}{}' \frac{1}{m^{t+s}} + O\left(\frac{\phi(q)}{q^s}\right) + O\left(\frac{\phi(q)}{q^s}\right)\\
&\quad + O\left(\frac{\phi(q)}{q^s}(\frac{a^{t-s}}{q^{t-s}}+\frac{a^t 2^s}{q^t})\right) + O\left(\frac{\phi(q)a^t}{q^t(q-a[\frac{q}{a}])^s}\right)\\
&= \frac{1}{2}\frac{\phi(q)}{a^s}\zeta(s+t)\prod_{p|q}(1-p^{-s-t}) + O\left(\frac{\phi(q)}{q^s}\right) + O\left(\frac{\phi(q)a^t}{q^t(q-a[\frac{q}{a}])^s}\right),
\end{aligned}$$

while in the case $a = 1$, the result holds with the last O-term not appearing. So that

$$\begin{aligned}
\sum_{\substack{a\leqslant N\\(a,k)=1}} M_1 &= \frac{\phi(q)}{2}\zeta(s+t)\prod_{p|q}(1-p^{-s-t})\sum_{\substack{a\leqslant N\\(a,k)=1}}\frac{1}{a^s} + O\left(\frac{\phi(q)}{q^s}\sum_{\substack{a\leqslant N\\(a,k)=1}} 1\right)\\
&\quad + O\left(\frac{\phi(q)}{q^t}\sum_{\substack{2\leqslant a\leqslant N\\(a,k)=1}}\frac{a^t}{(q-a[\frac{q}{a}])^s}\right). \qquad (2)
\end{aligned}$$

Note that

$$\sum_{\substack{a\leqslant N\\(a,k)=1}}\frac{1}{a^s} = \zeta(s)\prod_{p|k}(1-p^{-s}) + O\left(N^{-s+1}\right) \qquad (3)$$

and

$$\sum_{\substack{2\leqslant a\leqslant N\\(a,k)=1}}\frac{a^t}{\left(q-a[\frac{q}{a}]\right)^s} \ll N^t\sum_{u\leqslant q-1}\sum_{\substack{a\leqslant N\\ q-a[\frac{q}{a}]=u}}\frac{1}{u^s} \ll N^t\sum_{u\leqslant q-1}\frac{d(q-u)}{u^s} \ll N^t q^{\varepsilon}, \qquad (4)$$

where $d(q-u)$ is the divisor function. Inserting (3) and (4) into (2), we have

$$\begin{aligned}
\sum_{\substack{a\leqslant N\\(a,k)=1}} M_1 &= \frac{\phi(q)}{2}\zeta(s+t)\zeta(s)\prod_{p|q}(1-p^{-s-t})\prod_{p|k}(1-p^{-s})\\
&\quad + O\left(\frac{\phi(q)}{q^t}q^{\varepsilon}N^t\right) + O\left(\phi(q)N^{-s+1}\right). \qquad (5)
\end{aligned}$$

(ii)

$$\begin{aligned}
\sum_{\substack{a\leqslant N\\(a,k)=1}} M_2 &= \sum_{\substack{a\leqslant N\\(a,k)=1}} t \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\int_{\frac{q}{a}}^{+\infty}\frac{B(\bar{\chi},z)}{z^{t+1}}dz\right)\\
&= \sum_{\substack{a\leqslant N\\(a,k)=1}} t \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\int_{\frac{q}{a}}^{q}\frac{B(\bar{\chi},z)}{z^{t+1}}dz\right)\\
&\quad+ \sum_{\substack{a\leqslant N\\(a,k)=1}} t \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\int_{q}^{+\infty}\frac{B(\bar{\chi},z)}{z^{t+1}}dz\right)\\
&= \sum_{\substack{a\leqslant N\\(a,k)=1}} t \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\int_{\frac{q}{a}}^{q}\frac{B(\bar{\chi},z)}{z^{t+1}}dz\right)\\
&\quad+ \sum_{\substack{a\leqslant N\\(a,k)=1}} t\int_{q}^{+\infty}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)B(\bar{\chi},z)\right]\frac{1}{z^{t+1}}dz. \qquad (6)
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{\substack{a\leqslant N\\(a,k)=1}} t \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\left(\int_{\frac{q}{a}}^{q}\frac{B(\bar{\chi},z)}{z^{t+1}}dz\right)\\
\ll\ & \phi(q)\sum_{n\leqslant q}\frac{1}{n^s}\int_{\frac{q}{N}}^{q}\frac{1}{z^{t+1}}\left(\sum_{\substack{\frac{q}{z}<a\leqslant N\\(a,k)=1}}\ \sum_{\substack{\frac{q}{a}<m\leqslant z<q+\frac{q}{a}\\ n\equiv\pm ma \bmod q}} 1\right)dz\\
\ll\ & \phi(q)\sum_{n\leqslant q}\frac{1}{n^s}\int_{\frac{q}{N}}^{q}\frac{Nzq^{\varepsilon}}{qz^{t+1}}dz \ll \frac{\phi(q)}{q^t}N^t q^{\varepsilon}, \qquad (7)
\end{aligned}$$

where we have used the fact that for any fixed positive integers l and m, the number of the solutions of equation $an = lq + m$ (for all positive integers a and n) is $\ll q^{\varepsilon}$.

On the other hand,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)\sum_{\frac{q}{a}<m\leqslant z}\bar{\chi}(m) \ll \phi(q){\sum_{n\leqslant q}}'\ {\sum_{\substack{\frac{q}{a}<m\leqslant z<q+\frac{q}{a}\\ n\equiv\pm ma \bmod q}}}'\frac{1}{n^s} \ll \phi(q),$$

hence

$$\begin{aligned}
&\sum_{\substack{a\leqslant N\\(a,k)=1}} t\int_q^{+\infty}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(a)\left(\sum_{n\leqslant q}\frac{\chi(n)}{n^s}\right)B(\bar{\chi},z)\right]\frac{1}{z^{t+1}}dz\\
\ll\ &\sum_{\substack{a\leqslant N\\(a,k)=1}}\int_q^{+\infty}\frac{\phi(q)}{z^{t+1}}dz \ll \frac{\phi(q)}{q^t}N. \qquad (8)
\end{aligned}$$

With (7) and (8), we obtain

$$\sum_{\substack{a\leqslant N\\(a,k)=1}} M_2 \ll \frac{\phi(q)}{q^t}N^t q^{\varepsilon}. \qquad (9)$$

(iii) Changing the order of the summation and the integration implies that

$$M_3 = s\int_q^{+\infty}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(a)\left(\sum_{m\leqslant \frac{q}{a}}\frac{\bar{\chi}(m)}{m^t}\right)A(\chi,y)\right]\frac{1}{y^{s+1}}dy.$$

In order to estimate the integrand in M_3, we may replace $A(\chi,y)$ by $\sum\limits_{n\leqslant y<q}\chi(n)$ and get

$$\begin{aligned}
\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(a)\left(\sum_{m\leqslant \frac{q}{a}}\frac{\bar{\chi}(m)}{m^t}\right)\sum_{n\leqslant y<q}\chi(n) \ &\ll\ \phi(q)\sideset{}{'}\sum_{m\leqslant\frac{q}{a}}\ \sideset{}{'}\sum_{\substack{n<q\\ n\equiv -ma \bmod q}}\frac{1}{m^t}+\phi(q)\sideset{}{'}\sum_{m\leqslant\frac{q}{a}}\ \sideset{}{'}\sum_{\substack{n<q\\ n\equiv ma \bmod q}}\frac{1}{m^t}\\
&\ll\ \phi(q).
\end{aligned}$$

So that

$$\begin{aligned}
\sum_{\substack{a\leqslant N\\(a,k)=1}} M_3 \ &=\ s\sum_{\substack{a\leqslant N\\(a,k)=1}}\int_q^{+\infty}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(a)\left(\sum_{m\leqslant \frac{q}{a}}\frac{\bar{\chi}(m)}{m^t}\right)\sum_{n\leqslant y<q}\chi(n)\right]\frac{1}{y^{s+1}}dy\\
&\ll \sum_{\substack{a\leqslant N\\(a,k)=1}}\int_q^{+\infty}\frac{\phi(q)}{y^{s+1}}dy \ll \frac{\phi(q)}{q^s}N. \qquad (10)
\end{aligned}$$

(iv)

$$\sum_{\substack{a\leqslant N\\(a,k)=1}} M_4 \ =\ \sum_{\substack{a\leqslant N\\(a,k)=1}}\left[\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bar{\chi}(a)\left(s\int_q^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy\right)\left(t\int_{\frac{q}{a}}^{+\infty}\frac{B(\overline{\chi},z)}{z^{t+1}}dz\right)\right]$$

$$\ll \int_{\frac{q}{N}}^{+\infty}\int_{q}^{+\infty}\left(\sum_{\substack{\frac{q}{z}<a\leqslant N\\(a,k)=1}}\sum_{\frac{q}{a}<m<z}{}' \sum_{q<n<y}{}' \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(n)\overline{\chi}(ma)\right)\frac{1}{y^{s+1}}\frac{1}{z^{t+1}}dydz$$

$$\ll \phi(q)\int_{\frac{q}{N}}^{+\infty}\int_{q}^{+\infty}\left(\sum_{\substack{\frac{q}{z}<a\leqslant N\\(a,k)=1}}\sum_{\substack{\frac{q}{a}<m\leqslant\frac{q}{a}+q\\ n\equiv\pm ma \bmod q}}{}' \sum_{n\leqslant q}{}' 1\right)\frac{1}{y^{s+1}}\frac{1}{z^{t+1}}dydz$$

$$\ll \phi(q)\int_{\frac{q}{N}}^{+\infty}\int_{q}^{+\infty}qN\frac{1}{y^{s+1}}\frac{1}{z^{t+1}}dydz \ll \frac{\phi(q)}{q^{s+t-1}}N^{t+1}. \tag{11}$$

The lemma for $\lambda\equiv 1(\mathrm{mod}2)$ follows from (1), (5), (6), (10) and (11).

(2) If $\lambda\equiv 0(\mathrm{mod}2)$, χ is an even character mod q. Note that when $(q,mn)=1$,

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}\chi(n)\overline{\chi}(m)=\begin{cases}\frac{1}{2}\phi(q), & \text{if}\quad n\equiv\pm m \bmod q,\\ 0, & \text{otherwise.}\end{cases}$$

Using the same methods as previous, we can easily obtain the lemma for this case. This completes the proof of the lemma.

§3. Proof of the theorem

In this section, we will complete the proof of the theorem.

(i) If $m,\ n$ are odd numbers with $m,\ n>1$, we will get from 1) of Lemma 2.1 that

$$\begin{aligned}\sum_{h\leqslant N}{}' S(h,m,n,k) &= \sum_{h\leqslant N}{}'\left[\frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n}}\sum_{d|k}\frac{d^{m+n}}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\bar{\chi}(-h)L(m,\chi)L(n,\bar{\chi})\right]\\ &= \frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n+2}}\sum_{d|k}\frac{d^{m+n}}{\phi(d)}\sum_{h\leqslant N}{}'\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\bar{\chi}(h)L(m,\chi)L(n,\bar{\chi}).\end{aligned}$$

From Lemma 2.2, we can deduce that

$$\sum_{h\leqslant N}{}' S(h,m,n,k)$$

$$= \frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n+2}}\sum_{d|k}\frac{d^{m+n}}{\phi(d)}\left[\frac{\phi(d)}{2}\zeta(m+n)\zeta(m)\prod_{p|d}(1-p^{-m-n})\prod_{p|k}(1-p^{-m})\right.$$

$$
\begin{aligned}
&+ O\left(\phi(d)d^{-m+1}\left\{\frac{N}{d}\right\}^{-m+1} + \phi(d)d^{\varepsilon}\left\{\frac{N}{d}\right\}^{n}\right)\Bigg] \\
= \ & \frac{2m!n!}{k^{m+n-1}(2i\pi)^{m+n+2}}\zeta(m+n)\zeta(m)\sum_{d|k} d^{m+n}\prod_{p|d}(1-p^{-m-n})\prod_{p|k}(1-p^{-m}) \\
&+O\left(\frac{1}{k^{m+n-1}}\left[\sum_{d|k} d^{n+1}\left\{\frac{N}{d}\right\}^{-n+1} + \sum_{d|k} d^{m+n+\varepsilon}\left\{\frac{N}{d}\right\}^{n}\right]\right). \qquad (12)
\end{aligned}
$$

Note

$$
\sum_{d|k} d^{m+n}\prod_{p|d}(1-p^{-m-n}) = k^{m+n}, \qquad (13)
$$

$$
\begin{aligned}
\sum_{d|k} d^{n+1}\left\{\frac{N}{d}\right\}^{-m+1} &= \sum_{\substack{d|k\\ d>N}} d^{n+1}\left\{\frac{N}{d}\right\}^{-m+1} + \sum_{\substack{d|k\\ d<N}} d^{n+1}\left\{\frac{N}{d}\right\}^{-m+1} \\
&\ll \sum_{\substack{d|k\\ d>N}} d^{n+1}\left(\frac{N}{d}\right)^{-m+1} + \sum_{\substack{d|k\\ d<N}} d^{n+1}\left(\frac{1}{d}\right)^{-m+1} \\
&\ll N^{-m}k^{m+n+1+\varepsilon} + N^{m+n}k^{\varepsilon} \qquad (14)
\end{aligned}
$$

and

$$
\sum_{d|k} d^{m+n+\varepsilon}\left(\frac{N}{d}\right)^{n} \ll N^{n}k^{m+\varepsilon}, \qquad (15)
$$

the last O-term of (12) can be estimated as

$$
\begin{aligned}
&\frac{1}{k^{m+n-1}}\left[\sum_{d|k} d^{n+1}\left\{\frac{N}{d}\right\}^{-m+1} + \sum_{d|k} d^{m+n+\varepsilon}\left\{\frac{N}{d}\right\}^{n}\right] \\
\ll \ & N^{-m}k^{2+\varepsilon} + N^{m+n}k^{-m-n+1+\varepsilon} + N^{n}k^{-n+1+\varepsilon}. \qquad (16)
\end{aligned}
$$

Combining with (12) and (13), we get the first part of the theorem.

(ii) If $m,\ n$ are positive even numbers, we will get from (ii) of Lemma 2.1 that

$$
\begin{aligned}
&{\sum_{h\leqslant N}}' S(h,m,n,k) \\
= \ & {\sum_{h\leqslant N}}'\left[\frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n}}\sum_{d|k}\frac{d^{m+n}}{\phi(d)}\sum_{\chi \bmod d\chi(-1)=1}\bar{\chi}(-h)L(m,\chi)L(n,\bar{\chi})\right. \\
&\left. + \frac{4m!n!}{(2\pi)^{m+n}}\zeta(m)\zeta(n)\right] \\
= \ & \frac{4m!n!}{k^{m+n-1}(2i\pi)^{m+n}}\sum_{d|k}\frac{d^{m+n}}{\phi(d)}{\sum_{h\leqslant N}}'\sum_{\substack{\chi \bmod d\\ \chi(-1)=1}}\bar{\chi}(h)L(m,\chi)L(n,\bar{\chi}) + O(N). \qquad (17)
\end{aligned}
$$

Lemma 2.2 indicates that

$$\sum_{h\leqslant N}{}' S(h,m,n,k) = \frac{2m!n!}{k^{m+n-1}(2i\pi)^{m+n}}\zeta(m+n)\zeta(m)\sum_{d|k}d^{m+n}\prod_{p|d}(1-p^{-m+n})\prod_{p|k}(1-p^{-m})$$
$$+O\left(\frac{1}{k^{m+n-1}}\left[\sum_{d|k}d^{n+1}\left\{\frac{N}{d}\right\}^{-m+1}+\sum_{d|k}d^{m+n+\varepsilon}\left\{\frac{N}{d}\right\}^{n}\right]\right)+O(N).$$

We apply the same methods of (i) in the theorem to obtain (ii). This completes the proof of the theorem.

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Generalised Weyl and Weyl type theorems for algebraically k^*-paranormal operators

S. Panayappan[†], D. Sumathi[‡] and N. Jayanthi[♯]

Post Graduate and Research Department of Mathematics,
Government Arts College, Coimbatore 18, India
E-mail: Jayanthipadmanaban@yahoo.in

Abstract If λ is a nonzero isolated point of the spectrum of k^*-paranormal operator T for a positive integer k, then the Riesz idempotent operator E of T with respect to λ satisfies $E_\lambda H = \ker(T-\lambda) = \ker(T-\lambda)^*$ and E_λ is self-adjoint. We prove that if T is an algebraically k^*-paranormal operator for a positive integer k, then spectral mapping theorem and spectral mapping theorem for essential approximate point spectrum hold for T, Generalised Weyl's theorem holds for T and other Weyl type theorems are discussed.

Keywords k^*-paranormal, algebraically k^*-paranormal, Generalised Weyl's theorem, Polaroid.

§1. Introduction and preliminaries

Let $B(H)$ be the Banach algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator T, we mean an element in $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. The ascent of T denoted by $p(T)$, is the least non-negative integer n such that $\ker T^n = \ker T^{n+1}$. The descent of T denoted by $q(T)$ is the least non-negative integer n such that $ran(T^n) = ran(T^{n+1})$. T is said to be of finite ascent if $p(T-\lambda) < \infty$, for all $\lambda \in C$. If $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ ([11], Proposition 38.3), Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T.

An operator T is said to have the single valued extension property ($SVEP$) at $\lambda_0 \in C$, if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator T is said to have $SVEP$, if T has $SVEP$ at every point $\lambda \in C$. An operator T is called a Fredholm operator if the range of T denoted by $ran(T)$ is closed and both $\ker T$ and $\ker T^*$ are finite dimensional and is denoted by $T \in \Phi(H)$. An operator T is called upper semi-Fredholm operator, $T \in \Phi_+(H)$, if $ran(T)$ is closed and $\ker T$ is finite dimensional. An operator T is called lower semi- Fredholm operator, $T \in \Phi_-(H)$, if $\ker T^*$ is finite dimensional. The index of a semi-Fredholm operator is an integer defined as $ind(T) = \dim\ker T - \dim\ker T^*$. An upper semi-Fredholm operator, with index less than or equal to 0 is called upper semi-Weyl operator and is denoted by $T \in \Phi_+^-(H)$.

A lower semi-Fredholm operator with index greater than or equal to 0 is called lower semi-Weyl operator and is denoted by $T \in \Phi_{-}^{+}(H)$. A Fredholm operator of index 0 is called Weyl operator.

An upper semi-Fredholm operator with finite ascent is called upper semi-Browder operator and is denoted by $T \in B_{+}(H)$ while a lower semi-Fredholm operator with finite descent is called lower semi-Browder operator and is denoted by $T \in B_{-}(H)$. A Fredholm operator with finite ascent and descent is called Browder operator. Clearly, the class of all Browder operators is contained in the class of all Weyl operators. Similarly the class of all upper semi-Browder operators is contained in the class of all upper semi-Weyl operators and the class of all lower semi-Browder operators is contained in the class of all lower semi-Weyl operators. An operator T is Drazin invertible, if it has finite ascent and descent.

For an operator T and a non-negative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. In particular, $T_{[0]} = T$. If for some integer n, $R(T^n)$ is closed and $T_{[n]}$ is an upper(resp. a lower) semi-Fredholm operator, then T is called an upper(resp. lower) semi-B-Fredhom operator. Moreover if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. The index of a semi-B-Fredholm operator T is the index of semi-Fredholm operator $T_{[d]}$, where d is the degree of the stable iteration of T and defined as $d = \inf\{n \in N;$ for all $m \in N,\ m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T))\}$. T is called a B-Weyl operator if it is B-Fredholm of index 0.

The spectrum of T is denoted by $\sigma(T)$, where

$$\sigma(T) = \{\lambda \in C : T - \lambda I \text{ is not invertible}\}.$$

The approximate point spectrum of T is denoted by $\sigma_a(T)$, where

$$\sigma_a(T) = \{\lambda \in C : T - \lambda I \text{ is not bounded below}\}.$$

The essential spectrum of T is defined as

$$\sigma_e(T) = \{\lambda \in C : T - \lambda I \text{ is not Fredholm}\}.$$

The essential approximate point spectrum of T is defined as

$$\sigma_{ea}(T) = \{\lambda \in C : T - \lambda I \notin \Phi_{+}^{-}(H)\}.$$

The Weyl spectrum of T is defined as

$$w(T) = \{\lambda \in C : T - \lambda I \text{ is not Weyl}\}.$$

The Browder spectrum of T is defined as

$$\sigma_b(T) = \{\lambda \in C : T - \lambda I \text{ is not Browder}\}.$$

The set of all isolated eigenvalues of finite multiplicity of T is denoted by $\pi_{00}(T)$ and the set of all isolated eigenvalues of finite multiplicity of T in $\sigma_a(T)$ is denoted by $\pi_{00}^{a}(T)$. $p_{00}(T)$ is defined as $p_{00}(T) = \sigma(T) - \sigma_b(T)$. $E(T)$ denotes the isolated eigenvalues of T with no restriction on multiplicity.

The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in C : T - \lambda I \text{ is not a } B\text{-Weyl operator}\}.$$

We say that Weyl's theorem holds for T [8] if T satisfies the equality

$$\sigma(T) - w(T) = \pi_{00}(T)$$

and a-Weyl's theorem holds for T [17], if T satisfies the equality

$$\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}^a(T).$$

We say that T satisfies generalized Weyl's theorem [5] if

$$\sigma(T) - \sigma_{BW}(T) = E(T).$$

We say that T satisfies property (w) if

$$\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}(T)$$

and T satisfies property (b) if

$$\sigma_a(T) - \sigma_{ea}(T) = p_{00}(T).$$

By [7], if Generalized Weyl's theorem holds for T, then Weyl's theorem holds for T.

An operator T is called normaloid if $r(T) = \|T\|$, where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. An operator T is called hereditarily normaloid, if every part of it is normaloid.

An operator T is called polaroid if $iso\,\sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of poles of the resolvent of T and $iso\,\sigma(T)$ is the set of all isolated points of $\sigma(T)$. An operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator T is said to be reguloid if for every isolated point λ of $\sigma(T)$, $\lambda I - T$ is relatively regular. An operator T is known as relatively regular if and only if $\ker T$ and $T(X)$ are complemented. Also Polaroid $\Rightarrow$ reguloid $\Rightarrow$ isoloid.

K. S. Ryoo and P. Y. Sik defined k^*-paranormal operators in [18], k being a positive integer and showed that *-paranormal operators form a proper subclass of k^*-paranormal operators for $k \geq 3$, and k^*-paranormal operators are normaloid.

In this paper, we prove that k^*-paranormal operators have H property and if $0 \neq \lambda$ is an isolated point of the spectrum of k^*-paranormal operator T for a positive integer k, then the Riesz idempotent operator E of T with respect to λ satisfies $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$. We also show that if T is k^*-paranormal operator, then T is polaroid and Weyl's theorem hold for both T and T^*. If in addition T^* has $SVEP$, then a-Weyl's theorem hold for both T and T^* and also for $f(T)$ for every $f \in H(\sigma(T))$, the space of all analytic functions on an open neighborhood of spectrum of T.

We define algebraically k^*-paranormal operators and prove that if T is algebraically k^*-paranormal, then Generalised Weyl's theorem hold for T and Weyl's theorem hold for T and $f(T)$, for every $f \in H(\sigma(T))$, T is polaroid and hence has $SVEP$. We prove that if either T or T^* is algebraically k^*-paranormal, then spectral mapping theorem holds for essential approximate point spectrum of T. Other Weyl type theorems are also discussed.

§2. Definition and properties

In this section, we charecterise k^*-paranormal operators and using Matrix representation, we prove that the restriction of k^*-paranormal operators to an invariant subspace is also k^*-paranormal and $\ker(T-\lambda) \subset \ker(T-\lambda)^*$.

Definition 2.1.[18] An operator T is called k^*-paranormal for a positive integer k, if for every unit vector x in H, $\|T^k x\| \geq \|T^* x\|^k$.

Example 2.1. Let H be the direct sum of a denumerable number of copies of two dimensional Hilbert space $R \times R$. Let A and B be two positive operators on $R \times R$. For any fixed positive integer n, define an operator $T = T_{A,B,n}$ on H on as follows:

$$T((x_1, x_2, x_3, \cdots)) = (0, A(x_1), A(x_2), \cdots, A(x_n), B(x_{n+1}), \cdots).$$

Its adjoint T^* is given by

$$T^*((x_1, x_2, x_3, \cdots)) = (A(x_2), A(x_3), \cdots, A(x_n), B(x_{n+1}), \cdots).$$

Let A and B are positive operators satisfying $A^2 = C$ and $B^4 = D$, where $C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}$, then $T = T_{A,B,n}$ is of k^*-paranormal for $k = 1$.

Theorem 2.1.[18] For $k \geq 3$, there exists a k^*-paranormal operator which is not *-paranormal operator.

Theorem 2.2.[18] If T is a k^*-paranormal operator, then T is normaloid.

Theorem 2.3. An operator T is k^*-paranormal for a positive integer k if and only if for any $\mu > 0$,

$$T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0.$$

Proof. Let $\mu > 0$ and $x \in H$ with $\|x\| = 1$. Using arithmetic and geometric mean inequality, we get

$$\begin{aligned} \frac{1}{k}\left\langle \mu^{-k+1} \left|T^k\right|^2 x, x \right\rangle + \frac{k-1}{k}\langle \mu x, x\rangle &\geq \left\langle \mu^{-k+1} \left|T^k\right|^2 x, x \right\rangle^{\frac{1}{k}} \langle \mu x, x\rangle^{\frac{k-1}{k}} \\ &= \left\|T^k x\right\|^{\frac{2}{k}} \\ &\geq \|T^* x\|^2 \\ &= \langle TT^* x, x\rangle . \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\mu^{-k+1}}{k}\left\langle \left|T^k\right|^2 x, x \right\rangle + \frac{(k-1)\mu}{k}\langle x, x\rangle - \langle TT^* x, x\rangle \geq 0. \\ \Rightarrow \quad & T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0. \end{aligned}$$

Conversely assume that $T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k \geq 0$.

If $\|T^* x\| = 0$, then the k^*-paranormality condition is trivially satisfied.

If $x \in H$ with $\|T^*x\| \neq 0$ and $\|x\| = 1$, taking $\mu = \|T^*x\|^2$, we get

$$\left\|T^k x\right\| \geq \|T^*x\|^k.$$

Hence T is k^*-paranormal.

Theorem 2.4. If $T \in B(H)$ is a k^*-paranormal operator for a positive integer k, T does not have a dense range and T has the following representation: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T)} \oplus \ker(T^*)$, then T_1 is also a k^*-paranormal operator on $\overline{ran(T)}$ and $T_3 = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \bigcup \{0\}$, where $\sigma(T)$ denotes the spectrum of T.

Proof. Let P be the orthogonal projection onto $\overline{ran(T)}$. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$.

Since T is of k^*-paranormal operator, by Theorem 2.3,

$$P(T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k)P \geq 0.$$

Hence

$$T_1^{*k}T_1^k - k\mu^{k-1}(T_1T_1^* + T_2T_2^*) + (k-1)\mu^k \geq 0.$$

Hence

$$T_1^{*k}T_1^k - k\mu^{k-1}T_1T_1^* + (k-1)\mu^k \geq k\mu^{k-1}\left|T_2^*\right|^2 \geq 0.$$

Hence T_1 is also k^*-paranormal operator on $\overline{ran(T)}$.

Also for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\langle T_3x_2, x_2\rangle = \langle T(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^*(I-P)x\rangle = 0$. Hence $T_3 = 0$. By ([10], Corollary 7), $\sigma(T_1) \bigcup \sigma(T_3) = \sigma(T) \bigcup \tau$, where τ is the union of certain of the holes in $\sigma(T)$ which happen to be a subset of $\sigma(T_1) \bigcap \sigma(T_3)$, and $\sigma(T_1) \bigcap \sigma(T_3)$ has no interior points. Therefore $\sigma(T) = \sigma(T_1) \bigcup \sigma(T_3) = \sigma(T_1) \bigcup \{0\}$.

Theorem 2.5. If T is k^*-paranormal operator for a positive integer k and M is an ivariant subspace of T, then the restriction $T_{|M}$ is k^*-paranormal.

Proof. Let P be the orthogonal projection onto M. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$.

Since T is of k^*-paranormal operator, by Theorem 2.3,

$$P(T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k)P \geq 0.$$

Hence

$$T_1^{*k}T_1^k - k\mu^{k-1}(T_1T_1^* + T_2T_2^*) + (k-1)\mu^k \geq 0.$$

Hence

$$T_1^{*k}T_1^k - k\mu^{k-1}T_1T_1^* + (k-1)\mu^k \geq k\mu^{k-1}\left|T_2^*\right|^2 \geq 0.$$

Hence T_1, i.e., $T_{|M}$ is also k^*-paranormal operator on M.

Theorem 2.6. If T is k^*-paranormal operator for a positive integer k, $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$, then

(i) $T_2 = 0$,

(ii) T_3 is k^*-paranormal.

Proof. Let $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \ker(T-\lambda)^*$. Without loss of generality assume that $\lambda = 1$. Then by Theorem 2.3 for $\mu = 1$,

$$0 \leq T^{*k}T^k - k\mu^{k-1}TT^* + (k-1)\mu^k = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix},$$

where $X = -T_2T_2^*$, $Y = T_2 + T_2T_3 + \cdots + T_2T_3^{k-1} - kT_2T_3^*$ and $Z = Y^*Y + T_3^{*k}T_3^k - kT_3T_3^* + (k-1)$. A matrix of the form $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0,\ Z \geq 0$ and $Y = X^{1/2}WZ^{1/2}$, for some contraction W. Therefore $T_2T_2^* = 0$ and T_3 is k^*-paranormal.

Corollary 2.1. If T is k^*-paranormal operator for a positive integer k and $(T-\lambda)x = 0$ for $\lambda \neq 0$ and $x \in H$, then $(T-\lambda)^*x = 0$.

Corollary 2.2. If T is k^*-paranormal operator for a positive integer k, $0 \neq \lambda \in \sigma_p(T)$, then T is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \overline{ran(T-\lambda)^*}$, where T_3 is k^*-paranormal and $\ker(T_3 - \lambda) = \{0\}$.

§3. Spectral properties

If $\lambda \in iso\,\sigma(T)$, the spectral projection (or Riesz idempotent) E_λ of T with respect to λ is defined by $E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(z-T)^{-1}\,\mathrm{d}z$, where D is a closed disk with centre at λ and radius small enough such that $D \bigcap \sigma(T) = \{\lambda\}$. Then $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma(T_{|E_\lambda H}) = \{\lambda\}$ and $\ker(T-\lambda) \subset E_\lambda H$.

In this section, we show that k^*-paranormal operators have (H) property and if $\lambda \in \sigma(T)$ is an isolated point, then E_λ with respect to λ is self-adjoint and satisfies $E_\lambda H = \ker(T-\lambda) = \ker(T-\lambda)^*$. Weyl's theorem hold for both T and T^* and if T^* has $SVEP$, then a-Weyl's theorem hold for both T and T^*.

Theorem 3.1. If T is k^*-paranormal operator for a positive integer k and for $\lambda \in C$, $\sigma(T) = \lambda$ then $T = \lambda$.

Proof. If $\lambda = 0$, then since k^*-paranormal operators are normaloid ([18], Theorem 9), $T = 0$. Assume that $\lambda \neq 0$. Then T is an invertible normaloid operator with $\sigma(T) = \lambda$. $T_1 = \frac{1}{\lambda}T$ is an invertible normaloid operator with $\sigma(T_1) = \{1\}$. Hence T_1 is similar to an invertible isometry B (on an equivalent normed linear space) with $\sigma(B) = 1$ ([12], Theorem 2). T_1 and B being similar, 1 is an eigenvalue of $T_1 = \frac{1}{\lambda}T$ ([12], Theorem 5). Therefore by theorem 1.5.14 of [14], $T_1 = I$. Hence $T = \lambda$.

Theorem 3.2. If T is k^*-paranormal operator for some positive integer k, then T is polaroid.

Proof. If $\lambda \in iso\,\sigma(T)$ using the spectral projection of T with respect to λ, we can write $T = T_1 \oplus T_2$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Since T_1 is k^*-paranormal operator

and $\sigma(T_1) = \{\lambda\}$, by Theorem 3.1, $T_1 = \lambda$. Since $\lambda \notin \sigma(T_2)$, $T_2 - \lambda I$ is invertible. Hence both $T_1 - \lambda I$ and $T_2 - \lambda I$ and hence $T - \lambda I$ have finite ascent and descent. Hence λ is a pole of the resolvent of T. Hence T is polaroid.

Corollary 3.1. If T is k^*-paranormal operator for some positive integer k, then T is reguloid.

Corollary 3.2. If T is k^*-paranormal operator for some positive integer k, then T is isoloid.

Theorem 3.3. If T is k^*-paranormal operator for a positive integer k and $\lambda \in \sigma(T)$ is an isolated point, then the Riesz idempotent operator E_λ with respect to λ satisfies $E_\lambda H = \ker(T - \lambda)$. Hence λ is an eigenvalue of T.

Proof. Since $\ker(T - \lambda) \subseteq E_\lambda H$, it is enough to prove that $E_\lambda H \subseteq \ker(T - \lambda)$. Now $\sigma(T_{|E_\lambda H}) = \{\lambda\}$ and $T_{|E_\lambda H}$ is k^*-paranormal. Therefore by Theorem 3.1, $T_{|E_\lambda H} = \lambda$. Hence $E_\lambda H = \ker(T - \lambda)$.

Theorem 3.4. Let T be a k^*-paranormal operator for a positive integer k and $\lambda \neq 0$ be an isolated point in $\sigma(T)$. Then the Riesz idempotent operator E_λ with respect to λ is self-adjoint and satisfies $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$.

Proof. Without loss of generality assume that $\lambda = 1$. Let $T = \begin{pmatrix} 1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T - \lambda) \oplus \overline{ran(T - \lambda)^*}$. By Theorem 2.6, $T_2 = 0$ and T_3 is k^*-paranormal. Since $1 \in iso\sigma(T)$, either $1 \in iso\sigma(T_3)$ or $1 \notin \sigma(T_3)$. If $1 \in iso\sigma(T_3)$, since T_3 is isoloid, $1 \in \sigma_p(T_3)$ which contradicts $\ker(T_3 - \lambda) = \{0\}$ (by Corollary 2.2). Therefore $1 \notin \sigma(T_3)$ and hence $T_3 - 1$ is invertible. Therefore $T - 1 = 0 \oplus (T_3 - 1)$ is invertible on H and $\ker(T - 1) = \ker(T - 1)^*$. Also $E_\lambda = \frac{1}{2\pi i}\int_{\partial D}(zI - T)^{-1}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial D}\begin{pmatrix} (z-1)^{-1} & 0 \\ 0 & (z - T_3)^{-1} \end{pmatrix}\mathrm{d}z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore E_λ is the orthogonal projection onto $\ker(T - \lambda)$ and hence E_λ is self-adjoint.

Let $T \in L(X)$ be a bounded operator. T is said to have property (H) if $H_0(\lambda I - T) = \ker(\lambda I - T)$, where $H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}$. By [13], $E_\lambda H = H_0(\lambda I - T)$. Hence by Theorem 3.3, k^*-paranormal operators have (H) property. Hence by Theorems 2.5, 2.6 and 2.8 of [3], we get the following results:

Theorem 3.5. If T is k^*-paranormal operator for some positive integer k, then T has $SVEP$, $p(\lambda I - T) \leq 1$ for all $\lambda \in C$ and T^* is reguloid.

Theorem 3.6. If T is k^*-paranormal operator for some positive integer k, then Weyl's theorem holds for T and T^*. If in addition, T^* has $SVEP$, then a-Weyl's theorem holds for both T and T^*.

Theorem 3.7. If T is k^*-paranormal operator for some positive integer k and T^* has $SVEP$, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

§4. Algebraically k^*-paranormal operators

In this section, we prove spectral mapping theorem and essential approximate point spectral theorem for algebraically k^*-paranormal operators and also show that they are polaroids.

Definition 4.1. An operator T is defined to be of algebraically k^*-paranormal for a positive integer k, if there exists a non-constant complex polynomial $p(t)$ such that $p(T)$ is of class k^*-paranormal.

If T is algebraically k^*-paranormal operator for some positive integer k, then there exists a non-contant polynomial $p(t)$ such that $p(T)$ is k^*-paranormal. By the Theorem 3.5, $p(T)$ is of finite ascent. Hence $p(T)$ has $SVEP$ and hence T has $SVEP$ ([14], Theorem 3.3.6).

Theorem 4.1. If T is algebraically k^*-paranormal operator for some positive integer k and $\sigma(T) = \mu_0$, then $T - \mu_0$ is nilpotent.

Proof. Since T is algebraically k^*-paranormal there is a non-constant polynomial $p(t)$ such that $p(T)$ is k^*-paranormal for some positive integer k, then applying Theorem 3.1,

$$\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\} \text{ implies } p(T) = p(\mu_0).$$

Let $p(z) - p(\mu_0) = a(z-\mu_0)^{k_0}(z-\mu_1)^{k_1}\cdots(z-\mu_t)^{k_t}$ where $\mu_j \neq \mu_s$ for $j \neq s$.

Then $0 = p(T) - p(\mu_0) = a(T-\mu_0)^{k_0}(T-\mu_1)^{k_1}\cdots(T-\mu_t)^{k_t}$. Since $T-\mu_1$, $T-\mu_2$, $\cdots$, $T-\mu_t$ are invertible, $(T-\mu_0)^{k_0} = 0$. Hence $T - \mu_0$ is nilpotent.

Theorem 4.2. If T is algebraically k^*-paranormal operator for some positive integer k, then $w(f(T)) = f(w(T))$ for every $f \in Hol(\sigma(T))$.

Proof. Suppose that T is algebraically k^*-paranormal for some positive integer k, then T has $SVEP$. Hence by ([11], Proposition 38.5), $ind(T-\lambda) \leq 0$ for all complex numbers λ. Now to prove the result it is sufficient to show that $f(w(T)) \subseteq w(f(T))$. Let $\lambda \in f(w(T))$. Suppose if $\lambda \notin w(f(T))$, then $f(T) - \lambda I$ is Weyl and hence $ind(f(T) - \lambda) = 0$. Let $f(z) - \lambda = (z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n)g(z)$. Then $f(T) - \lambda = (T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n)g(T)$ and $ind(f(T)-\lambda) = 0 = ind(T-\lambda_1) + ind(T-\lambda_2) + \cdots + ind(T-\lambda_n) + ind g(T)$. Since each of $ind(T-\lambda_i) \leq 0$, we get that $ind(T-\lambda_i) = 0$, for all $i = 1, 2, \cdots, n$. Therefore $T - \lambda_i$ is weyl for each $i = 1, 2, \cdots, n$. Hence $\lambda_i \notin w(T)$ and hence $\lambda \notin f(w(T))$, which is a contradiction. Hence the theorem.

Theorem 4.3. If T or T^* is algebraically k^*-paranormal operator for some positive integer k, then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

Proof. For $T \in B(H)$, by [16] the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in H(\sigma(T))$ with no restrictions on T. Therefore, it is enough to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.

Suppose if $\lambda \notin \sigma_{ea}(f(T))$ then $f(T) - \lambda \in \Phi_+^-(H)$, that is $f(T) - \lambda$ is upper semi-Fredholm operator with index less than or equal to zero. Also $f(T) - \lambda = c(T-\alpha_1)(T-\alpha_2)\cdots(T-\alpha_n)g(T)$, where $g(T)$ is invertible and c, α_1, $\alpha_2, \cdots, \alpha_n \in C$.

If T is algebraically k^*-paranormal for some positive integer k, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is k^*-paranormal. Then $p(T)$ has $SVEP$ and hence T has $SVEP$. Therefore $ind(T-\alpha_i) \leq 0$ and hence $T - \alpha_i \in \Phi_+^-(H)$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

If T^* is algebraically k^*-paranormal for some positive integer k, then there exists a non-constant polynomial $p(t)$ such that $p(T^*)$ is k^*-paranormal. Then $p(T^*)$ has $SVEP$ and hence T^* has $SVEP$. Therefore $ind(T-\alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$. Therefore $0 \leq \Sigma_{i=1}^n ind(T - \alpha_i) = ind(f(T) - \lambda) \leq 0$. Therefore $ind(T-\alpha_i) = 0$ for each $i = 1, 2, \cdots, n$. Therefore

$T-\alpha_i$ is Weyl for each $i=1,2,\cdots,n$. $(T-\alpha_i)\in\Phi_+^-(H)$ and hence $\alpha_i\notin\sigma_{ea}(T)$. Therefore $\lambda=f(\alpha_i)\notin f\left(\sigma_{ea}(T)\right)$. Hence $\sigma_{ea}\left(f(T)\right)=f\left(\sigma_{ea}(T)\right)$.

Theorem 4.4. If T is algebraically k^*-paranormal operator for some positive integer k, then T is polaroid.

Proof. If $\lambda\in iso\,\sigma(T)$ using the spectral projection of T with respect to λ, we can write $T=T_1\oplus T_2$ where $\sigma(T_1)=\{\lambda\}$ and $\sigma(T_2)=\sigma(T)-\{\lambda\}$. Since T_1 is algebraically k^*-paranormal operator and $\sigma(T_1)=\{\lambda\}$, by Theorem 4.1, $T_1-\lambda I$ is nilpotent. Since $\lambda\notin\sigma(T_2)$, $T_2-\lambda I$ is invertible. Hence both $T_1-\lambda I$ and $T_2-\lambda I$ and hence $T-\lambda I$ have finite ascent and descent. Hence λ is a pole of the resolvent of T. Hence T is polaroid.

Corollary 4.1. If T is algebraically k^*-paranormal operator for some positive integer k, then T is reguloid.

Corollary 4.2. If T is algebraically k^*-paranormal operator for some positive integer k, then T is isoloid.

§5. Generalised Weyl's theorem and other Weyl type theorems

In this section, we prove Generalised Weyl's theorem for algebraically k^*-paranormal operators and discuss other Weyl type theorems.

Theorem 5.1. If T is algebraically k^*-paranormal operator for some positive integer k, then generalized Weyl's theorem holds for T.

Proof. Assume that $\lambda\in\sigma(T)-\sigma_{BW}(T)$, then $T-\lambda$ is B-Weyl and not invertible. Claim: $\lambda\in\partial\sigma(T)$. Assume the contrary that λ is an interior point of $\sigma(T)$. Then there exists a neighborhood U of λ such that $\dim N(T-\mu)>0$ for all μ in U. Hence by ([9], Theorem 10), T does not have $SVEP$ which is a contradiction. Hence $\lambda\in\partial\sigma(T)-\sigma_{BW}(T)$. Therefore by punctured neighborhood theorem, $\lambda\in E(T)$.

Conversely suppose that $\lambda\in E(T)$. Then λ is isolated in $\sigma(T)$. Using the Riesz idempotent E_λ with respect to λ, we can represent T as the direct sum $T=\begin{pmatrix} T_1 & 0\\ 0 & T_2\end{pmatrix}$ where $\sigma(T_1)=\{\lambda\}$ and $\sigma(T_2)=\sigma(T)-\{\lambda\}$. Then by Theorem 4.1, $T_1-\lambda$ is nilpotent. Since $\lambda\notin\sigma(T_2)$, $T_2-\lambda$ is invertible. Hence both $T_1-\lambda$ and $T_2-\lambda$ have both finite ascent and descent. Hence $T-\lambda$ has both finite ascent and descent. Hence $T-\lambda$ is Drazin invertible. Therefore, by ([6], Lemma 4.1), $T-\lambda$ is B-Fredholm of index 0. Hence $\lambda\in\sigma(T)-\sigma_{BW}(T)$. Therefore $\sigma(T)-\sigma_{BW}(T)=E(T)$.

Corollary 5.1. If T is algebraically k^*-paranormal operator for some positive integer k, then Weyl's theorem holds for T.

By ([4], Theorem 2.16) we get the following result:

Corollary 5.2. If T is algebraically k^*-paranormal for some positive integer k and T^* has $SVEP$ then a-Weyl's theorem and property(w) hold for T.

Theorem 5.2. If T is algebraically k^*-paranormal operator for some positive integer k, then Weyl's theorem holds for $f(T)$, for every $f\in Hol(\sigma(T))$.

Proof. For every $f \in H(\sigma(T))$,

$$\begin{aligned}\sigma(f(T)) - \pi_{00}(f(T)) &= f(\sigma(T) - \pi_{00}(T)) \qquad \text{by ([15], Lemma)}\\ &= f(w(T)) \qquad \text{by Theorem 5.2.}\\ &= w(f(T)) \qquad \text{by Theorem 4.3.}\end{aligned}$$

Hence Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.

If T^* has $SVEP$, then by ([1], Lemma 2.15), $\sigma_{ea}(T) = w(T)$ and $\sigma(T) = \sigma_a(T)$. Hence we get the following results:

Corollary 5.3. If T is algebraically k^*-paranormal for some positive integer k and if in addition T^* has $SVEP$, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Corollary 5.4. If T^* is algebraically k^*-paranormal for some positive integer k, then $w\left(f(T)\right) = f\left(w(T)\right)$.

By ([1], Theorem 2.17), we get the following results:

Corollary 5.5. If T is algebraically k^*-paranormal for some positive integer k and T^* has $SVEP$ then property (b) hold for T.

Corollary 5.6. If T is algebraically k^*-paranormal for some positive integer k, Weyl's theorem, a-Weyl's theorem, property (w) and property (b) hold for T^*.

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